Discrete harmonic function in $\mathbb{Z}^n$

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Abstract

This is a note after reading paper [1]

1 The statement of result

First of all, we give the definition of discrete harmonic function.

Definition 1 (Discrete harmonic function) *We say a function* $f : \mathbb{Z}^n \to \mathbb{R}$ *is a discrete harmonic function on* $\mathbb{Z}^n$ *if and only if for any* $(x_1, ..., x_n) \in \mathbb{Z}^n$, *we have:*

$$f(x_1, ..., x_n) = \frac{1}{2^n} \sum_{(\delta_1,...,\delta_n)\in\{-1,1\}^n} f(x_1 + \delta_1, ..., x_n + \delta_n) \tag{1}$$

In dimension 2, the definition reduce to:

Definition 2 (Discrete harmonic function in $\mathbb{R}^2$) *We say a function* $f : \mathbb{Z}^2 \to \mathbb{R}$ *is a discrete harmonic function on* $\mathbb{Z}^2$ *if and only if for any* $(x_1, x_2) \in \mathbb{Z}^2$, *we have:*

$$f(x_1, x_2) = \frac{1}{4} \sum_{(\delta_1,\delta_2)\in\{-1,1\}^2} f(x_1 + \delta_1, x_2 + \delta_2) \tag{2}$$

The result establish in [1] is following:

Theorem 3 (Liouville theorem for discrete harmonic functions in $\mathbb{R}^2$) *Given* $c > 0$. *There exists a constant* $\epsilon > 0$ *related to* c *such that, given a discrete harmonic function* f *in* $\mathbb{Z}^2$ *satisfied for any ball* $B_R(x_0)$ *with radius* $R > R_0$, *there is* $1 - \epsilon$ *portion of points* $x \in B_R(x_0)$ *satisfied* $|f(x)| < c$. *then* f *is a constant function in* $\mathbb{Z}^2$.

Remark 4 *This type of result contradict to the intuition, at least there is no such result in* $\mathbb{C}$. *For example. the existence of poisson kernel and the example given in [1] explain the issue.*

Remark 5 *There are reasons to explain why there could not have a result in* $\mathbb{C}$ *but in* $\mathbb{Z}^2$*,*

1. *The first reason is due to every radius* R *there is only* $O(R^2)$ *lattices in* $B_R(x)$ *in* $\mathbb{Z}^2$ *so the mass could not concentrate very much in this setting.*
2. *The second one is due to there do not have infinite scale in* $\mathbb{Z}^2$ *but in* $\mathbb{C}$*.*
3. *The third one is the function in* $\mathbb{Z}^2$ *is automatically locally integrable.*

The generation is following:

Theorem 6 (Liouville theorem for discrete harmonic functions in $\mathbb{R}^n$**)** *Given* $c > 0, n \in \mathbb{N}$*. There exists a constant* $\epsilon > 0$ *related to* n, c *such that, given a discrete harmonic function* f *in* $\mathbb{Z}^n$ *satisfied for any ball* $B_R(x_0)$ *with radius* $R > R_0$*, there is* $1 - \epsilon$ *portion of points* $x \in B_R(x_0)$ *satisfied* $|f(x)| < c$*. then* f *is a constant function in* $\mathbb{Z}^n$*.*

In this note, I give a proof of 6, and explicit calculate a constant $\epsilon_n > 0$ satisfied the condition in 3, this way could also calculate a constant ϵ_n satisfied 6. and point the constant calculate in this way is not optimal both in high dimension and 2 dimension.

2 some element properties with discrete harmonic function

We warm up with some naive property with discrete harmonic function. The behaviour of bad points could be controlled, just by isoperimetric inequality and maximum principle we have following result.

Definition 7 (Bad points) *We divide points of* $\mathbb{Z}^n$ *into good part and bad part, good part* I *is combine by all point* x *such that* $|f(x)| < c$*, and* J *is the residue one. So* $A \amalg B = \mathbb{Z}^n$*.*

For all $B_R(0)$*, we define* $J_R := J \cap B_R(0), I_R = I \cap B_R(0)$ *for convenient.*

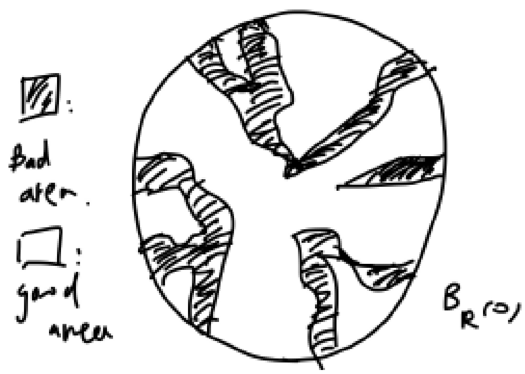
Theorem 8 (The distribution of bad points) *For all bad points* J_R *in* $B_R(0)$*, they will divide into several connected part, i.e.*

$$J_R = \amalg_{i \in S_R} A_i \tag{3}$$

and every part A_i *satisfied* $A_i \cap \partial B_R(0) \neq \emptyset$*.*

Remark 9 *We say* A *is connected in* $\mathbb{Z}^n$ *iff there is a path in* A *connected* $x \to y, \forall x, y \in A$*.*

Remark 10 *the meaning that every point So the behaviour of bad points are just like a tree structure given in the gragh.*



of bad points.png

Proof: A very naive observation is that for all $\Omega \subset \mathbb{Z}^n$ is a connected compact domain, then there is a function

$$\lambda_\Omega : \partial\Omega \times \mathring{\Omega} \longrightarrow \mathbb{R} \tag{4}$$

such that $\lambda_\Omega(x, y) \geq 0, \forall (x, y) \in \Omega \times \mathring{\Omega}$. And we have:

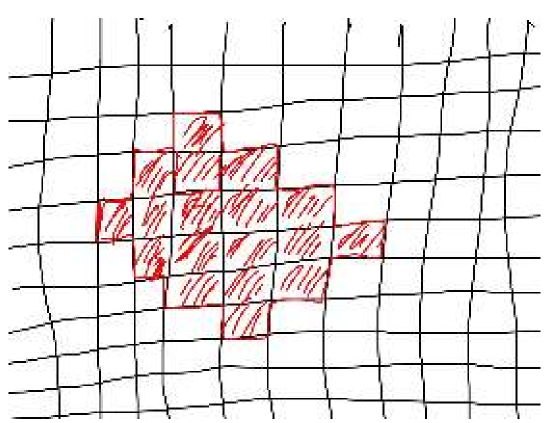
$$f(x) = \sum_{y \in \partial\Omega} \lambda_\Omega(x, y)(y) \tag{5}$$

This could be proved by induction on the diameter if Ω.

Then, if there is a connected component of Ω such that contradict to theorem 8 for simplify assume the connected component is just Ω, then use the formula 5we know

$$\begin{aligned} \sup_{x\in\Omega} |f(x)| &= \sup_{x\in\Omega} \sum_{y\in\partial\Omega} \lambda_\Omega(x, y)(y) \\ &\leq \sup_{\partial\Omega} |f(x)| \\ &\leq c \end{aligned}$$

The last line is due to consider around $\partial\Omega$. But this lead to: $\forall x \in \Omega, |f(x)| < c$ which is contradict to the definition of Ω. So we get the proof. □

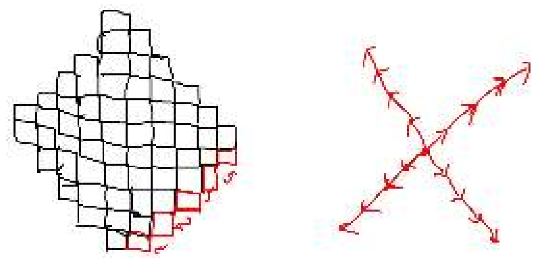


of the bad points.png

Now we begin another observation, that is the freedom of extension of discrete harmonic function in $\mathbb{Z}^n$ is limited.

Theorem 11 *we can say something about the structure of harmonic function space of Z^n, the cube, you will see, if add one value, then you get every value, i.e. we know the generation space of Z^n*

PROOF: For two dimension case, the proof is directly induce by the diameter of Ω. The case of n dimensional is similar. □



of extension .png

Remark 12 *The generation space is well controlled. In fact is just like n orthogonal direction line in n dimensional case.*

3 sktech of the proof for 6

The proof is following, by looking at the following two different lemmas establish by two different ways, and get a contradiction.

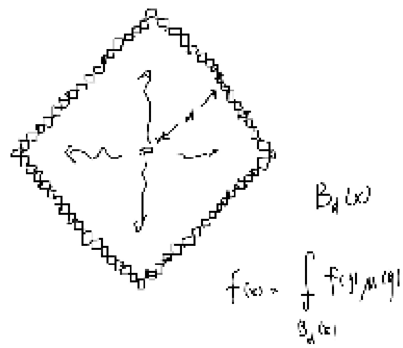
First lemma

Lemma 13 (Discrete poisson kernel) *the poisson kernel in $\mathbb{Z}^n$. We point out there is a discrete poisson kernel in $\mathbb{Z}^n$, this is given by:*

$$f(x) = \sum_{y \in \partial B_R(z)} \lambda_{B_R(z)}(x, y)(y) \tag{6}$$

And the following properties is true:

1. $\lambda_{B_R(z+h)}(x + h, y + h) = \lambda_{B_R(z)}(x, y)$, $\forall x \in \Omega, h \in \mathbb{Z}^n$.
2. $$\lambda_{B_R(z)}(x, y) \to \rho_R(x, y) \tag{7}$$



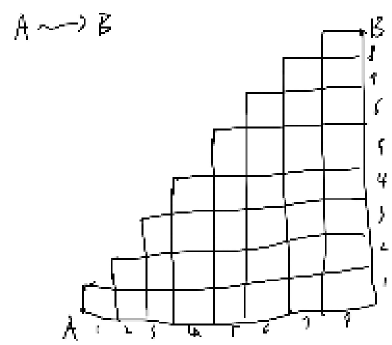
value property.png

Remark 14 *The proof could establish by central limit theorem, brown motion, see the material in the book of Stein [2]. The key point why this lemma 13 will be useful for the proof is due to this identity always true $\forall x \in B_R(0)$, So we will gain a lots of identity, These identity carry information which is contract by another argument.*

Second lemma The exponent decrease of mass.

Lemma 15 *The mass decrease at least for exponent rate.*

Remark 16 *the proof reduce to a random walk result and a careful look at level set, reduce to the worst case by brunn-minkowski inequality or isoperimetry inequality.*



of path form A to B.png

Final argument By looking at lemma 1 and lemma 2, we will get a contradiction by following way, first the value of f on $\partial B_R(0)$ increasing too fast, exponent increasing by lemma2, but on the other hand, it lie in the integral expresion involve with poisson kernel, but the pertubation of poisson kernel is slow, polynomial rate in fact...

References

[1] A DISCRETE HARMONIC FUNCTION BOUNDED ON A LARGE PORTION OF Z2 IS CONSTANT 1

[2] Functional analysis 5