Dirichlet hyperbola method

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January 8, 2018

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1 Introduction

Theorem 1

$$\sum_{1\leq n\leq x} d(n) = \sum_{1\leq n\leq x} [\frac{x}{n}] = xlogx + (2\gamma - 1)x + O(\sqrt{x}) \tag{1}$$

Remark 2 *I thought this problem initial 5 years ago, cost me several days to find a answer, I definitely get something without the argument of Dirchlet hyperbola method and which is weaker but morally the same camparable with the result get by Dirichlet hyperbola method.*

Remark 3 *How to get the formula:*

$$\sum_{1\leq n\leq x} d(x) = \sum_{1\leq n\leq x} [\frac{x}{n}]? \tag{2}$$

In fact,

$$\sum_{1\leq n\leq x} d(x) = \sum_{1\leq ab\leq x} 1 = \sum_{1\leq n\leq x} [\frac{x}{n}] \tag{3}$$

Which is the integer lattices under or lying on the hyperbola $\{(a,b)|ab = x\}$*.*

Remark 4 *By trivial argument, we can bound the quantity as following way,*

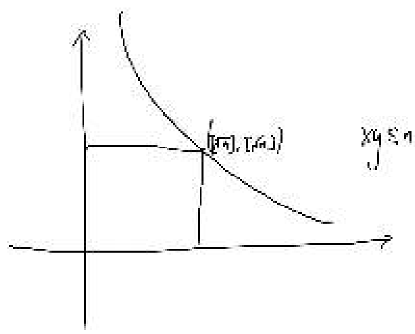
$$\begin{aligned}
\sum_{1\leq n\leq x} [\frac{x}{n}] &= \sum_{1\leq ab\leq x} 1 \\
&= x\sum_{i=1}^{x} \frac{1}{i} - \sum_{i=1}^{x} \{\frac{x}{i}\} \\
&= xlnx + \gamma x + O(x)
\end{aligned}$$

The error term is $O(x)$, which is too big. But fortunately we can use the symmetry of hyperbola to improve the error term.

PROOF:

$$\begin{aligned}\sum_{1\le n\le x} d(n) &= \sum_{ab\le x} 1 \\ &= \sum_{a\ge\sqrt{x}} [\frac{x}{b}] + \sum_{b\ge\sqrt{x}} [\frac{x}{a}] - \sum_{1\le a,b\le\sqrt{x}} 1 \\ &= xlogx + (2\gamma - 1)x + O(\sqrt{x})\end{aligned}$$

□



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Theorem 5 *Given a natural number k, use the hyperbola method together with induction and partial summation to show that*

$$\sum_{n\le x} d_k(n) = xP_k(logx) + O(x^{1-\frac{1}{k}+\epsilon}), n \le x \tag{4}$$

where $P_k(t)$ denotes a polynomial of degree $k-1$ with leading term $\frac{t^{k-1}}{(k-1)!}$.

Remark 6 *$P_k(x)$ is the residue of $\zeta(s)^k x^s s^{-1}$ at $s = 1$.*

PROOF:

We can establish the dimension 3 case directly, which is the following asymptotic formula,

$$\sum_{1\le xy\le n} [\frac{n}{xy}] = xP_2(logx) + O(x^{1-\frac{1}{3}+\epsilon}) \tag{5}$$

The approach is following, we first observe that

$$\sum_{1\le xy\le n} [\frac{n}{xy}] = \sum_{xyz\le n} 1 \tag{6}$$

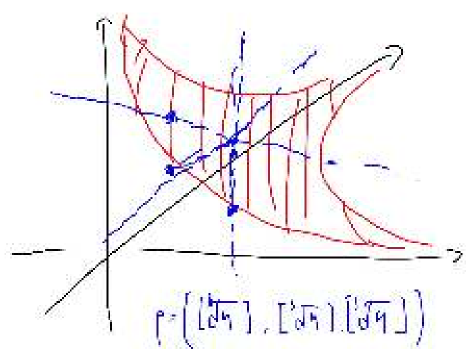
The problem transform to get a asymptotic formula for the lattices under 3 dimension hyperbola. The first key point is, morally $([n^{\frac{1}{3}}],[n^{\frac{1}{3}}],[n^{\frac{1}{3}}])$ is the central point under the hyperbola.
Then we can divide the range into 3 parts, and try to get a asymptotic formula for each part then add them together. Assume we have:

1. $A_x = \sum_{1\le r\le [n^{\frac{1}{3}}]} \sum_{1\le yz\le [n^{\frac{2}{3}}]} [\frac{r}{yz}]$.
2. $A_y = \sum_{1\le r\le [n^{\frac{1}{3}}]} \sum_{1\le xz\le [n^{\frac{2}{3}}]} [\frac{r}{yz}]$.
3. $A_z = \sum_{1\le r\le [n^{\frac{1}{3}}]} \sum_{1\le xy\le [n^{\frac{2}{3}}]} [\frac{r}{yz}]$.

Then the task transform to get a asymptotic formula,

$$A_x = A_y = A_z = xQ_2(logx) + O(x^{1-\frac{1}{3}+\epsilon}) \tag{7}$$

But we can do the same thing for $\sum_{1\le yz\le [n^{\frac{2}{3}}]} [\frac{r}{yz}]$ and then integral it. This end the proof. For general $k \in \mathbb{N}$, the story is the same, by induction.
Induction on k and use the Fubini theorem to calculate $\sum_{x_1\ldots x_r\le n} \frac{n}{x_1\ldots x_r}, \forall 1\le r\le k$.
□



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There is a major unsolved problem called Dirichlet divisor problem.

Problem 1

$$\sum_{n\le x} d(n) \tag{8}$$

What is the error term? The conjecture is the error term is $O(x^{\theta}), \forall\theta > \frac{1}{4}$, it is known that $\theta = \frac{1}{4}$ is not right.

Remark 7 *To beats this problem, need some tools in algebraic geometry.*

2 Several problems

Problem 2 *$\forall k \in \mathbb{N}$, is there a asymptotic formula for $\sum_{t=1}^{n}\{\frac{kn}{t}\}$?*

Problem 3 *$\forall k \in \mathbb{N}$, $f(n)$ is a polynomial with degree k, is there a asymptotic formula for $\sum_{t=1}^{n}\{\frac{f(n)}{t}\}$?*

Problem 4 *$\forall k \in \mathbb{N}$, $g(n)$ is a polynomial with degree k, is there a asymptotic formula for $\sum_{t=1}^{n}\{\frac{n}{g(t)}\}$?*

Theorem 8 *$k \in \mathbb{N}$, then we have*

$$\lim_{n\to\infty}\frac{\{\frac{kn}{1}\}+\{\frac{kn}{2}\}+...+\{\frac{kn}{n}\}}{n} = k(\sum_{i=1}^{k}\frac{1}{i} - lnk - \gamma) \tag{9}$$

PROOF:

$$\begin{aligned}\frac{\{\frac{kn}{1}\}+\{\frac{kn}{2}\}+...+\{\frac{kn}{n}\}}{n} &= \frac{\sum_{i=1}^{k}\frac{kn}{i} - \sum_{i=1}^{n}[\frac{kn}{i}]}{n}\\ &= k(lnn+\gamma+\epsilon_n) - \frac{\sum_{i=1}^{kn}[\frac{kn}{i}] - \sum_{i=n+1}^{kn}[\frac{kn}{i}]}{n}\end{aligned}$$

□ Now we try to estimate

$$S_k(n) = \sum_{i=1}^{kn}[\frac{kn}{i}] - \sum_{i=n+1}^{kn}[\frac{kn}{i}] \tag{10}$$

In fact, we have,

$$\begin{aligned}S_k(n) &= (2\sum_{i=1}^{[\sqrt{kn}]}[\frac{kn}{i}] - [\sqrt{kn}]^2) - (\sum_{i=1}^{k}[\frac{kn}{i}] - kn)\\ &= 2\sum_{i=1}^{[\sqrt{kn}]}\frac{kn}{i} - \sum_{i=1}^{k}\frac{kn}{i} + 2\{\sqrt{kn}\}[\sqrt{kn}] + \{\sqrt{kn}\}^2 - 2\sum_{i=1}^{[\sqrt{kn}]}\{\frac{kn}{i}\} + \sum_{i=1}^{k}\{\frac{kn}{i}\}\\ &= 2kn(ln[\sqrt{kn}] + \gamma + \epsilon_{[\sqrt{kn}]}) - kn\sum_{i=1}^{k}\frac{1}{i} + r(n)\\ &= knln(kn) + kn(2\gamma - \sum_{i=1}^{k}\frac{1}{i}) + r'(n)\\ &= knlnn + kn(2\gamma + lnk - \sum_{i=1}^{k}\frac{1}{i}) + r'(n)\end{aligned}$$

Where $-3\sqrt{n} < r(n) < 3\sqrt{n}$, $-3\sqrt{n} < r'(n) < 3\sqrt{n}$.

So by **??** we know,

$$\begin{aligned}\frac{\{\frac{kn}{1}\} + ... + \{\frac{kn}{n}\}}{n} &= k(lnn + \gamma + \epsilon_n) - klnn - k(2\gamma + lnk - \sum_{i=1}^{k} \frac{1}{i}) + \frac{r'(n)}{n} \\ &= k(\sum_{i=1}^{k} \frac{1}{i} - lnk - \gamma) + \frac{r'(n)}{n} + k\epsilon_n\end{aligned}$$

So we have,

$$\lim_{n\to\infty} \frac{\{\frac{kn}{1}\} + ... + \{\frac{kn}{n}\}}{n} = k(\sum_{i=1}^{k} \frac{1}{i} - lnk - \gamma) = k\epsilon_k \tag{11}$$

Remark 9 *In fact we can get* $0 < k\epsilon_k < \frac{1}{2}, \forall k \in \mathbb{N}$, *by combining the theorem* **??** *and* **??**.

3 Lattice points in ball

Gauss use the cube packing circle get the trivial estimate,

$$\sum_{n\leq x} r_2(n) = \pi x + O(\sqrt{x}) \tag{12}$$

In the same way one can obtain,

$$\sum_{n\leq x} r_k(n) = \rho_k x^{\frac{k}{2}} + O(x^{\frac{k-1}{2}}) \tag{13}$$

Remark 10 *Where* $\rho_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$ *is the volume of the unit ball in* k *dimension.*

Dirchlet's hyperbola method works nicely for the lattic points in a ball of dimension $k \geq 4$. Langrange proved that every natural number can be represented as the sum of four squares, i.e. $r_4(n) > 0$, and Jacobi established the exact formula for the number of representations

$$r_4(n) = 8(2 + (-1)^n) \sum_{d|n, d\ odd} d. \tag{14}$$

Hence we derive,

$$\begin{aligned}\sum_{n\leq x} r_4(n) &= 8\sum_{m\leq x}(2+(-1)^m)\sum_{dm\leq x, d\ odd} d \\ &= 8\sum_{m\leq x}(2+(-1)^m)(\frac{x^2}{4m^2}+O(\frac{x}{m})) \\ &= 2x^2\sum_{1}^{\infty}(2+(-1)^m)m^{-2}+O(xlogx) \\ &= 3\zeta(2)x^2+O(xlogx)=\frac{1}{2}(\pi x)^2+O(xlogx)\end{aligned}$$

This result extend easily for any $k \geq 4$, write r_k as the additive convolution of r_4 and r_{k-4}, i.e.

$$r_k(n)=\sum_{0\leq t\leq n} r_4(t)r_{k-4}(n-t) \tag{15}$$

Apply the above result for r_4 and execute the summation over the remaining $k-4$ squares by integration.

$$\sum_{n\leq x} r_k(n)=\frac{(\pi x)^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}+O(x^{\frac{k}{2}-1}logx) \tag{16}$$

Remark 11 *Notice that this improve the formula* **??** *which was obtained by the method of packing with a unit square. The exponent* $\frac{k}{2}-1$ *in* **??** *is the best possible because the individual terms of summation can be as large as the error term (apart from logx), indeed for* $k=4$ *we have* $r_4(n)\geq 16n$ *if* n *is odd by the Jacobi formula.*

The only case of the lattice point problem for a ball which is not yet solved (i.e. the best possible error terms are not yet established) are for the circle($k=2$*) and the sphere (*$k=3$*).*

Theorem 12

$$\sum_{n\leq x}\tau(n^2+1)=\frac{3}{\pi}xlogx+O(x) \tag{17}$$

4 Application in finite fields

Suppose $f(x)\in\mathbb{Z}[x]$ is a irreducible polynomial. And for each prime p, let

$$\rho_f(p)=\#\ of\ solutions\ of f(x)\equiv 0(mod\ p) \tag{18}$$

By Langrange theorem we know $\rho_f(p)\leq deg(f)$.

Problem 5 *Is there a asymptotic formula for*

$$\sum_{p\leq x}\rho_f(p)? \tag{19}$$

A general version, we can naturally generated it to algebraic variety.

$$\rho_{f_1,\ldots,f_k}(p) = \#\ of\ solutions\ of f_i(x) \equiv 0 (mod\ p),\ \forall 1 \leq i \leq k \tag{20}$$

Problem 6 *Is there a asymptotic formula for*

$$\sum_{p\leq x}\rho_{f_1,\ldots,f_k}(p)? \tag{21}$$

Example 1 *We give an example to observe what is involved.*
$f(x) = x^2 + 1$. *We know* $x^2 + 1 \equiv 0(mod\ p)$ *is solvable iff* $p \equiv 1(mod\ 4)$ *or* $p = 2$. *One side is easy, just by Fermat little theorem, the other hand need Fermat descent procedure, which of course could be done by Willson theorem. In this case,*

$$\sum_{p\leq n}\rho_f(p) = \#\{number\ of\ primes\ of\ type\ 4k+1\ in\ 1,2,\ldots,n\} \tag{22}$$

Which is a special case of Dirichlet prime theorem.

Let K be an algebraic number field, i.e. the finite field extension of rational numbers, let

$$\mathcal{O}_K = \{\alpha \in K, \alpha\ satisfied\ a\ monic\ polynomial\ in\ \mathbb{Z}[x]\} \tag{23}$$

Dedekind proved that,

Theorem 13

1. *$\mathcal{O}_K$ is a ring, we call it the ring of integer of K.*
2. *He showed further every non-zero ideal of $\mathcal{O}_K$ could write as the product of prime ideal in $\mathcal{O}_k$ uniquely.*
3. *the index of every non-zero ideal I in $\mathcal{O}_K$ is finite, i.e. $[\mathcal{O}_K : I] < \infty$, and we can define the norm induce by index.*

   $$N(I) := [\mathcal{O}_K : I] \tag{24}$$

   Then the norm is a multiplication function in the space of ideal, i.e. $N(IJ) = N(I)N(J), \forall I, J \in$ ideal class group of $\mathcal{O}_K$.
4. *Now he construct the Dedekind Riemann zeta function,*

   $$\zeta_K(s) = \sum_{N(I)\neq 0}\frac{1}{N(I)^s} = \prod_{J\ prime\ ideal}\frac{1}{1-\frac{1}{N(J)^s}},\ \forall Re(s) > 1 \tag{25}$$

Now we consider the analog of the prime number theorem.

Problem 7 *Let* $\pi_K(x) = \{I, N(I) < x\}$*, does the exist a asymptotic formula,*

$$\pi_K(x) \sim \frac{x}{lnx} \text{ as } x \to \infty? \tag{26}$$

Given a prime p, we may consider the prime ideal

$$p\mathcal{O}_K = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}...\mathfrak{P}_k^{e_k} \tag{27}$$

Where $\mathfrak{P}_i$ is different prime ideal in $\mathcal{O}_K$. But the question is how to find these $\mathfrak{P}_i$? For the question, there is a satisfied answer.

Lemma 14 (existence of primitive element) *There always exist a primetive elements in* K*, such that,*

$$K = \mathbb{Q}(\theta) \tag{28}$$

Where θ *is some algebraic number, which's minor polynomial* $f(x) \in \mathbb{Z}[x]$.

Theorem 15 (Dedekind recipe) *Take the polynomial* $f(x)$*, factorize it in the polynomial ring* $\mathbb{Z}_p[x]$*,*

$$f(x) \equiv f_1(x)^{e_1}...f_r(x)^{e_r}(mod\ p) \tag{29}$$

Consider $\mathfrak{P}_i = (p, f_i(\theta)) \subset \mathcal{O}_K$*. Then apart from finite many primes, we have,*

$$p\mathcal{O}_K = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}...\mathfrak{P}_k^{e_k} \tag{30}$$

Where $N(\mathfrak{P}_i) = p^{degf_i}$.

Remark 16 *The apart primes are those divide the discriminant.*

Now we can argue that **??** is morally the same as counting the ideals whose norm is divide by p in a certain algebraic number theory.

And we have following, which is just the version in algebraic number fields of **??**.

Theorem 17 (Weber) # *of ideals of* $\mathcal{O}_K$ *with norm* $\leq x$ *equal to,*

$$\rho_k(X) + O(x^{1-\frac{1}{d}}), where\ d = [K : Q] \tag{31}$$