Symplectic geometry

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1 Introduction

This is the first note of a series of notes concert on semiclassical analysis. Given the basic material on symplectic geometry. Including the following material,

1. The case at a point, or we can look it as the case in $\mathbb{R}^{2n}$.
2. The standard material in symplectic geometry, i.e. Hamiltonian mechanics, two approach, global one concentrating on lie derivative, and a locally one concentrating on the power of Darboux theorem, i.e. the existence of a canonical coordinate.
3. The basic facts on Poission bracket.
4. The basic facts on Lagrange sub-manifold, and the involve of Liouville measure.

2 Case of a point, or $\mathbb{R}^{2n}$

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, at once we have a vector field, we could consider the associated flow of it,

$$\begin{cases} \dot{\omega} = & V(\omega) \\ \omega(0) = & z \end{cases} \tag{1}$$

express the trajectory start from z along the vector field.

Remark 1 *There $\dot{\omega} := \frac{\partial \omega}{\partial t}$. One the other hand, due to the locally existence theorem of ODE, if the regularity of V is enough, then the solution exist and is uniqueness.*

Definition 2 *$\psi_t z = \omega(t, z)$ or more convenient $\psi_t := exp(tV)$. We call $\{\psi_t\}_{t\in\mathbb{R}}$ the flow map or the exponential map generated by V.*

Lemma 3 *For flow map, we have following:*

1. $\psi_0 z = z$ *for all* $z \in \mathbb{R}^N$.
2. $\psi_{t+s} = \phi_t \psi_s$ *for all* $s, t \in \mathbb{R}$.
3. *for each time* $t \in \mathbb{R}$, *the mapping* $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ *is a diffeomorphism with* $(\psi_t)^{-1} = \psi_{-t}$

So it is a group action on $\mathbb{R}^n$, with units as diffeomorphism of $\mathbb{R}^n$. PROOF:
This lemma is the direct corollary of the theory of ODE. □

Now let us special to the case $\mathbb{R}^{2N} = \mathbb{R}^n \times \mathbb{R}^n$. In local coordinate we have $z = (x, \xi)$, $x \in \mathbb{R}^n$ express position of particle, $\xi \in \mathbb{R}^n$ express momentum of particle.

Definition 4 $z = (x, \xi), w = (y, \eta)$ *in* $\mathbb{R}^{2n}$ *define their symplectic product,*

$$\sigma(z, w) := < \xi, y > - < x, \eta > \tag{2}$$

In a matrix form, σ *coincide with a* $2n \times 2n$ *matrix*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{3}$$

Following lemma given the basic property of σ, J.

Lemma 5 *The following basic property are true.*

1. $\sigma(z, \omega) = < Jz, w >$, $\forall z, w \in \mathbb{R}^{2n}$.
2. *the bilinear form* σ *is antisymmetric,* $\sigma(z, w) = -\sigma(w, z)$ *and degenerate, i.e. if* $\sigma(z, w) = 0$ *for all* $w \in \mathbb{R}^{2n}$, *then* $z = 0$.
3. $J^2 = -I$, $J^T = -J = J^{-1}$.

PROOF:

1. trivial calculate get this.
2. trivial.
3. $JJ^{-1} = -I$, by basic linear algebra everything follows.

□

3 Hamiltonian mechanics

Definition 6 *Symplectic form: non-degenerate closed 2 form in a standard coordinate(Darboux coordinate, coordinate like $\mathbb{R}^{2n}$) looks like,*

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{4}$$

$\forall x \in X$, map

$$T_x X \to T_x^* X$$

$$v \to w(\cdot, v)$$

is an isomorphism. w is called the symplectic form.

There is locally coordinate for TM, T^*M, i.e.,

$$TM : \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$$

$$T^*M : dx_1, ..., dx_n$$

So $w = f^{ij} dx_i \wedge dx_j$, roughly we have $w(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}) = f^{ik} - f^{ki}$, this is of course not true, but morally true. Now let us give the definition of symplectic manifold and the relationship of Hamiltonian mechanics.

Definition 7 *We have the following definition,*

1. *A symplectic manifold is a pair (X, ω) where X is a smooth manifold and w is a closed two-form on X such that $\forall x \in X$ the map,*

   $$T_x X \to T_x^* X$$

   $$v \to w(\cdot, v)$$

   is an isomorphism, w is called the symplectic form.

2. *If (X, w) is symplectic, and $f : X \to \mathbb{R}$ is differentiable, the hamiltonian vector field of f is the field Ξ_f on X whose image under the previous map is df. In other word, Ξ_f is characticed by the property,*

   $$w(\cdot, \Xi_f) = df(\cdot)$$

3. *The flow of Ξ_f will be referenced to as the hamiltonian flow of f.*

Lemma 8 *If $X = R^{2n}$ coordinate $(x_1, ..., x_n, p_1, ..., p_n) = (x, p)$ and the symplectic form*

$$w = \sum_{j=1}^{n} dp_j \wedge dx_j$$

then,

1. *If $f : \mathbb{R}^{2n} \to \mathbb{R}$ is differentiable, then the integral curves of Ξ_f are the solutions to the system of ODEs,*

$$\begin{cases} \dot{x}_j = & \dfrac{\partial f}{\partial p_j} \quad j = 1, 2, ..., n \\ \dot{p}_j = & -\dfrac{\partial f}{\partial x_j} \quad j = 1, 2, ..., n \end{cases} \tag{5}$$

2. *Moreover, if*

$$f(x, p) = \frac{1}{2m}\|p\|^2 + V(x). \tag{6}$$

Where V is a smooth solution ("potential"), and $(x(t), p(t))$ is a trajectory of the Hamiltonian flow of f, then

$$m\ddot{x} = -\nabla \tag{7}$$

This is the Newton's second law for the force $-\nabla V$.

Proof: $w = \sum_{j=1}^{n} dp_j \wedge dx_j$, due to we have $w(\cdot, \Xi_f =) = df$. So $\forall j$ we have:

$$w(\frac{\partial}{\partial p_j}, \Xi_f) = df(\frac{\partial}{\partial p_j}) = \frac{\partial f}{\partial p_j} \tag{8}$$

Assume $\Xi_f = \lambda^i \frac{\partial}{\partial x_i} + \gamma^i \frac{\partial}{\partial p_i}$. Then we have,

$$\begin{aligned} \frac{\partial f}{\partial p_j} &= w(\frac{\partial}{\partial p_j}, \lambda^i \frac{\partial}{\partial x_i} + \gamma^i \frac{\partial}{\partial p^i}) \\ &= \lambda^j \cdot (-1)^{\sigma(...)} \end{aligned}$$

and also,

$$\begin{aligned} \frac{\partial f}{\partial x_j} &= w(\frac{\partial}{\partial p_j}, \lambda^i \frac{\partial}{\partial x_i} + \gamma^i \frac{\partial}{\partial p^i}) \\ &= \gamma^j \cdot (-1)^{\sigma(...)} \end{aligned}$$

Combine with the definition of integral curve we derive the integral curves of Ξ_f i.e. $\gamma(t)$ such that $\dot{\gamma}(t) = \Xi_f$, $\gamma(t) = (x_1(t), ..., p_n(t))$ is given by 5.

Now we begin to proof the Newton second law for the force $-\nabla V$. We consider the 2-dimensional case at first. We have,

$$\begin{aligned}
m\ddot{x} &= m(\dot{\frac{\partial f}{\partial p}}) \\
&= m\frac{\partial}{\partial t}(\frac{\partial f}{\partial p}) \\
&= m\frac{\partial}{\partial p}[\frac{\partial}{\partial t}(\frac{\|p\|^2}{m} + V(x))] \\
&= m\frac{\partial}{\partial p}(\frac{2p\dot{p}}{m} + \frac{\partial V(x)}{\partial x}\cdot\frac{\partial f}{\partial p}), \mathit{due\ to}\ -\frac{\partial f}{\partial x} = -\frac{\partial V(x)}{\partial x}. \\
&= m\frac{\partial}{\partial p}(\frac{2p(-\frac{f}{\partial x})}{m} + \frac{\partial V(x)}{\partial x}\cdot\frac{\partial f}{\partial p}) \\
&= -\frac{\partial V(x)}{\partial x}
\end{aligned}$$

The high dimension case is similar, thanks to the linearity of V and $\ddot{x}$.

□

Remark 9 *Two make Newton's second law to be true, the form 6 play a crucial role. Is there some generalization of this type of result to more general case, roughly speaking, it is reasonable to expect this could still be true if the hamiltonian function could be divide into potential energy part and kinetic energy part. And the describe of potential energy part is that it is given by a quadratic form.*

Lemma 10 *In general, for any Hamilton field Ξ_f one has:*

1. $\pounds_{\Xi_f} f = 0$*, conservation of energy. In orther word, Ξ_f is everywhere tangent to the level sets of f.*
2. $\pounds_{\Xi_f}\omega = 0$*, so the Hamiltonian flow of f consists of automorphism of M, ω.*

General speaking, to proof a theorem on manifold, there always have two choice, coordinate free proof and proof in a careful choose coordinate. If we choose to believe the Darboux theorem 12 is true, the meaning of it is that locally the symplectic manifold are the same.

Proof: If we believe the Darboux theorem 12 is true. then consider in a standard coordinate $(x_1, ..., x_n, p_1, ..., p_n)$, we have,

$$\Xi_f = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i} - \sum_i \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i} \tag{9}$$

So of course $\pounds_{\Xi_f} f = \Xi f = 0$. In general case, i.e. coordinate free proof, $\omega(\cdot, \Xi_f) = df$. Use identity if lie derivative. The second thing is also easy to proof by look in a local canonical coordinate, involve the indentity of lie derivative. □

Remark 11 *I need more understanding on the lie derivative, see wiki.*

Theorem 12 *(Darboux theorem) Near any point there exist coordinate:*$(x_1, ..., x_n, p_1, ..., p_n)$ *usually called Darboux coordinates, such that the sympletic form* w *has the form,*

$$w = \sum_{j=1}^{n} dp_j \wedge dx_j. \tag{10}$$

Remark 13 *This theorem means there do not exist local invariant in symplectic manifold.*

PROOF:

□

Theorem 14 *If M is any smooth manifold, then its cotangent bundle $X = T^*M$ has a natural symplectic structure.*

Proof: we have local coordinate on T^M derive from $(x_1, ..., x_n, dx_1, ..., dx_n)$, it is $(x_1, ..., x_n, p_1, ..., p_n)$. Remember we have Riemann metric: $g^{ij}dx_i dx_j$ on $T^*M \otimes T^*M$, the existence of Riemann metric involve a unit decomposition argument and bump function, I just recall it there. Now we move on, consider the relationship between M, TM, T^*M.

$$M \longleftrightarrow T^*M \overset{pairing,(X,f)=X(f)}{\longleftrightarrow} TM \tag{11}$$

□

Remark 15 *It need not be the case that α non-degenerate $\Longrightarrow$ $d\alpha$ non-degenerate. This case in the lemma is a example to show that could be the case: α degenerate $\Longrightarrow$ $d\alpha$ non-degenerate.*

We glue something together on the space pf differential operator to understand the topology of it but not deifferential structure or more refinement structure.

Quntalization could be look as a way to glue, this could be down if there is a differential equation with some special condition (come from a flow take charge of it suffice).

Lemma 16 *(The proof of $d\alpha$ is non-degenerate) Let $(x_1, ..., x_n)$ be local coordinates on $U \subset M$. Define a coordinate system $(x_1, ..., x_n, p_1, ..., p_n)$ on T^*U by the condition:*

$$\forall \xi \in T_x^*U, p_j(\xi) = \xi(\frac{\partial}{\partial x_j}) \tag{12}$$

Prove that in Darboux coordinate, $\alpha = \sum_{j=1}^n p_j dx_j$ and therefore $\omega = \sum_{j=1}^n dp_j \wedge dx_j$.

Proof:

□

Theorem 17 *Let (M, g) be a smooth Riemann manifold and let $f : T^*M \to \mathbb{R}$ be one half of the square of the Riemann norm, so that in local coordinate,*

$$f(x, p) = \frac{1}{2} \sum_{i,j} g^{ij} p_i p_j \tag{13}$$

Then the trajectorics of the hamiltonian flow of f, projected down to M, are geodesic aries in this fashion.

Newton's second law+ energy vanish.

Proof:

$$m\ddot{x} = \nabla V = 0$$

So second variation formula describe of geodesic give us the fact that the trajective is geodesic. □

Remark 18 *We could directly calculate in local coordinate.*

4 Poisson brackets

f is the Hamiltonian generating the dynamic g is any smooth function on phase space (the symplectic manifold), then the rate of change of g along the trajectraries of d is the function

$$\dot{g} = \pounds_{\Xi_f} g = d_g(\Xi_f) = \omega(\Xi_f, \Xi_g) \tag{14}$$

Definition 19 *If (X, ω) is symplectic and $f, g \in C^\infty(X)$, the poisson bracket of f and g is defined to be the function on X.*

$$\{f, g\} := \omega(\Xi_f, \Xi_g) \tag{15}$$

Lemma 20 *In canonical (Darboux) coordinate where $\omega = \sum_j dp_j \wedge dx_j$, one has,*

$$\{f, g\} = \sum_{j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} \tag{16}$$

In particular, $\{p_j, x_j\} = \delta_{ij}$.

Proof:

$$\begin{aligned}
\{f, g\} &= \omega(\Xi_f, \Xi_g) \\
&= \sum_{j=1}^{n} dp_j \wedge dx_j \left(-\frac{\partial f}{\partial p_j}\frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}\frac{\partial}{\partial p_j}, -\frac{\partial g}{\partial p_j}\frac{\partial}{\partial x_j} + \frac{\partial g}{\partial x_j}\frac{\partial}{\partial p_j}\right) \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}.
\end{aligned}$$

□

Theorem 21 *If (X, ω) is a symplectic manifold then $\{C^\infty(X), \{\cdot\}\}$ is a Lie algebra.*

PROOF: Bilinearty, skew-symmetric come form,

$$\{f, g\} = \omega(\Xi_f, \Xi_g) \tag{17}$$

Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \tag{18}$$

could be proved by calculate under a local coordinate. □