Pesudo differential opertor and singular integral

XIYU HU

December 26, 2017

1 Introduction

There is two space to understand a function's behaviour, the physics space and the frequency space (Why thing going like this? Why there is such a duality?). Namely, we have:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi x} f(x) dx \tag{1}$$

The key point is, waves is a parameter group of scaling of definition of a constant fraquence wave, so it connected the multiplication and addition. Basically due to it can be look as the correlation of a function and the scaling of wave with carry all the information about f. A generation of this obeservation is the wavelet theory.

So as we well know, the key ingredient of Fourier transform is to image function as a sum of series waves. A famous theorem of Mikhlion said that a translation-invariant operator T on R^n could be represented by a multiplication operator on the Fourier transform side. translation is the meaning, $h \circ T = T \circ h, \forall h$ is a translation.

In a formal level, consider it as distribution (compact distribution or temperature distribution is both OK). We have:

$$T(e^{2\pi i x\xi}) = a(\xi) e^{2\pi i x\xi}, \forall \xi \in \mathbb{R}^n \tag{2}$$

the meaning is if we consider T is a operator on distribution space, $T : S' \to S'$, then $\forall f \in S$,

$$\int T(e^{2\pi i x\xi}) f = \int a(\xi) e^{2\pi i x\xi} f$$

due to the linear combination of $e^{2\pi i x\xi}$ will consititue a dense set in S. So this could extend to the whole distribution space by dual and give the definition of T, i.e.

$$(Tf)(x) = \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i x\xi} \hat{f}(\xi) d\xi \tag{3}$$

Remark 1 *T is bounded on $L^2(\mathbb{R}^n)$ when a is a bounded function, thanks to Parevel theorem. When a is a bounded function, the composition of two such operator could be defined, and the symbol of composition operator corresponding to the composite of their symbol, i.e.*

$$T_a \circ T_b(e^{2\pi i x\xi}) = b(\xi)a(\xi)e^{2\pi i x\xi} \tag{4}$$

Remark 2 *For parenval theorem, i.e. $\|\hat{f}\|_2 = \|f\|_2$, there is two approach, heat kernel approximation approach and discretization.*

We wish to investigate the operator given by multiplier, i.e.

$$(Tf)(x) = \int_{\mathbb{R}^n} a(x,\xi)e^{2\pi i x\xi}\hat{f}(\xi)d\xi \tag{5}$$

When it is satisfied $\|T\|_{p-p} < \infty$?

Intuition, the following calculate is only morally true, not rigorous.

$$\begin{aligned}
\|Tf\|_p^p &= \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} a(x,\xi)e^{2\pi i x\xi}\hat{f}(\xi)d\xi|^p dx \\
&\overset{\exists f}{\sim} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |a(x,\xi)e^{2\pi i x\xi}\hat{f}(\xi)|^p d\xi dx \\
&\overset{Fubini}{\sim} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\widehat{a(x,\xi)}\hat{f}(\xi)|^p dx d\xi \\
&\sim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |a(x,\xi)^\vee * f(\xi)|^p dx d\xi
\end{aligned}$$

So we need some restriction on $a(x,\xi)^\vee$, namely $\widehat{a(x,\xi)}$, so we need some decay condition on $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|$, why this, just consider integral by part for $a(x,\xi) \in S$. The rigorozaton of this intuition inspirit us to the definition of symbol calss.

Definition 3 *we say $a(x,\xi)$ is in symbol class S_m iff,*

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le A_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{6}$$

for all α, β is multi-indece.

Remark 4 *1. we note that all partial differential operator, whose coefficient, together with all their derivatives are bounded belong to this class, In this particular circumstance, the symbol is a polynomial in ξ, essentially the "characteristic polynomial" of the operator.*

2. *The general operator of this class have a parallel description in terms of their kernels. That is, in a suitable sense,*

$$(Tf)(x) = \int_{\mathbb{R}_x} K(x,y)f(y)dy \tag{7}$$

besides enjoying a cancellation property, K is here characterized by differential inequalities "dual" to those for $a(x,\xi)$. In the key case where the order $m = 0$, this kernel representation makes T a singular integral operator.

3. *The crucial L^2 estimate, when $m = 0$, is atelatively simple consequences of Plancherel's theorem for the Fourier transform. With this, the L^p theory introduce in previous note is therefore applicable.*

4. *The product identity that holds in the translation-invariant case generalized to the situation treated here as a symbolic calculus for the composition of operators. That is, there is an asymptotic formula for the composition of two such operators, whose main term is the point-wise product of their symbols.*

5. *The succeeding terms of the formula are of decreasing orders. These orders measure not only the size of the symbols, but determine also the increasing smoothing properties of the corresponding operators. The smoothing properties are most neatly expressed in terms of the Sobolev space W_k^p and the Lipschitz space Λ_α.*

2 Pseudo-differential operator

"Freezing principle": from variable coefficient differential equation to constant coefficient differential equation by approximation. divide into 2 steps:

1. divide space into small cubes.

2. take average of the coefficient of differential equation in every cubes.

Suppose we are interested in study the solution of the classical elliptic second order equation.

$$(Lu)(x) = \sum a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) \tag{8}$$

Where the coefficient matrix $\{a_{ij}(x)\}$ is assume to be real, symmetric, positive definite and smooth in X. Understanding P, such that,

$$PL = I \tag{9}$$

Looking for a P. Such that $PL = I + E$. E is a error term which have good control. To do this, fix an arbituary point x_0, freeze the operator L at x_0:

$$L_{x_0} = \sum a_{ij}(x_0) \frac{\partial^2}{\partial x_i \partial x_j} \tag{10}$$

In Fourier sense (L^2 sence).

$$\begin{aligned} L_{x_0} f(x) &= \int e^{2\pi i x \xi} (\sum \widehat{a_{ij}(x_0) \frac{\partial^2 f(\xi)}{\partial x_i \partial x_j}}) d\xi \\ &= \int e^{2\pi i x \xi} \int e^{-2\pi i \xi y} \sum a_{ij}(x_0) \frac{\partial^2}{\partial x_i \partial x_j} f(y) dy d\xi \\ &= \int e^{2\pi i x \xi} (-4\pi^2) \sum_{i,j} a_{ij}(x_0) \xi_i \xi_j \end{aligned}$$

Remark 5 *The remark is, morally speaking, for application of fourier transform in PDE. morally we could only solve the problem with linear differential equation (although we could consider the hyperbolic type). The main obstacle for Fourier transform application into PDE:*

1. *it only make sense with Schwarz class or its dual, this is not main obstacle, in principle could be solved by rescaling.*
2. *the main obstacle is it only compatible with linear differential equation.*

Cut-off function: η vanish near the origin,

$$(P_{x_0} f) = \int_{\mathbb{R}^n} (-4\pi^2 \sum_{i,j} a_{ij}(x_0) \xi_i \xi_j)^{-1} \eta(\xi) \hat{f(\xi)} e^{2\pi i x \xi} d\xi. \tag{11}$$

then:

$$L_{x_0} P_{x_0} = I + E_{x_0}.$$

E_{x_0} is actually a smoothing operator, because it is given by convolution with a fixed test function. It should be seasonable when x near x_0, $(Pf)(x)$ is well approximated by $(P_{x_0} f)(x)$, it is actually the case, define $((Pf)(x) := (P_x f)(x)$, i.e.

$$(Pf)(x) = \int_{\mathbb{R}^n} (-4\pi^2 \sum_{i,j} a_{ij}(x) \xi_i \xi_j)^{-1} \eta(\xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi \tag{12}$$

The operator P so given is a propotype of a pesudo-differential operator. Moreover, one has $LP = I + E$, where the error operator E is "smoothing of order 1". That this is indeed the case is the main part of the symbolic calculus described.

Definition 6 *(symbol class) A function $a(x,\xi)$ belong to S^m and is said to be of order m of $a(x,\xi)$ is a C^∞ function of $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and satisfies the differential inequality:*

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{13}$$

For all α, β are multi-indece.

Now we trun to the exact meaning of pesudo-differential operator, i.e. how them action on functions. Under some suffice given regularity condition, for $a \in S^m$, $T_a : S \to S$.

$$(Tf)(x) = \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi \tag{14}$$

Remark 7 *$T_a : S \to S$ is continuous and for $a_k \to a$ pointwise, $a_k \in S, \forall k \in \mathbb{N}^*$, $T_{a_k}(f) \to T_a(f)$ in S.*

then expense it, we get:

$$(T_a f)(x) = \int\int a(x,\xi)e^{2\pi i\xi(x-y)}f(y)dyd\xi \tag{15}$$

This could be diverge, even when $f \in S$. The key point is we do not have control with the second integral, morally speaking, this phenomenon is the weakness of Lesbegue integral which would not happen in Riemann integral, so sometime we need the idea from Riemann integral, this phnomenon is settle by multi a cut off function η_ϵ and take $\epsilon \to \infty$, the same deal also occur as the introduced of P.V. integral in Hilbert transform. The precise method to deal with the obstacle is following: $a_\epsilon(x,\xi) = a(x,\xi)\gamma(\epsilon x, \epsilon\xi)$, if $a \in S^m$, $a_\epsilon \in S^m$. $T_{a_\epsilon} \to T_a$ in the sense:

$\forall f \in S$, $T_{a_\epsilon}(f) \to T_a(f)$,

$$(T_a f)(x) = \lim_{\epsilon\to 0}\int\int a_\epsilon(x,\xi)e^{2\pi i\xi(x-y)}f(y)dyd\xi \tag{16}$$

We also have:

$$< T_a f, g > = < f, T_a^* g >, \forall f, g \in S. \tag{17}$$

Then we have:

$$(T_a^* g)(y) = \lim_{\epsilon\to 0}\int\int \bar{a}_\epsilon(x,\xi)e^{2\pi i\xi(y-x)}g(x)dxd\xi \tag{18}$$

and $< f, g >$ denotes $\int_{\mathbb{R}^n} f(x)\bar{g}(x)dx$. Thus the pesudo-differential operator T_a initially defined as a mapping from S to S, extend via the identity 17 to a mapping from the space of temperatured distribution S' to itself S'. Notice also that T_a is automatically continuous in this space.

3 L^p bounded theorem

We first introduce a powerful tools, called dyadic decomposition,

Lemma 8 *(dyadic decomposition) In eculid space $\mathbb{R}^n$ there exists a function $\phi \in C^\infty(\mathbb{R}^n)$ such that,*

$$\sum_{i\in\mathbb{Z}} \phi(2^{-i}x) = 1 \tag{19}$$

and $\forall x \in \mathbb{R}^n$, there is only two of $i \in \mathbb{Z}$ such that $\phi(2^{-i}x) \neq 0$, and we can choose ϕ to be radical and $\phi(x) \geq 0, \forall x \in \mathbb{R}^n$.

Remark 9 *So for a given mutiplier $a(x,\xi)$, we will have $a(x,\xi) = \sum_{i\in\mathbb{Z}} a_i(x,\xi) = \sum_{i\in Z} \phi(2^{-ix})a(x)$.*

Proof: The proof is easy, after rescaling we just need observed there is a bump function satisfied whole condition. □

Theorem 10 *Suppose a is a symbol of order 0, i.e. that $a \in S^0$ Then the operator T_a, initially defined on S, extends to a bounded operator from $L^2(\mathbb{R}^n)$ to itself.*

Remark 11 *Suffice to show $\|T_a(f)\|_{L^2} \leq A\|f\|_{L^2}, \forall f \in S$ and by dual.*

In fact we can directly proof a more general theorem:

Theorem 12 *Let $m : \mathbb{R}^d - \{0\} \to \mathbb{C}$ satisfy, for any multi-index γ of length $|\gamma| \leq d+2$,*

$$|\partial^\gamma m(\xi)| \leq B|\xi|^{-|\gamma|}$$

For all $\xi \neq 0$. Then, for any $0 < p < \infty$, there is a constant $C = C(d,p)$ such that,

$$\|(m\hat{f})^\vee\|_p \leq C(p,d)\|f\|_p \tag{20}$$

for all $f \in S$.

Proof: $a \in S^0$, so we have:

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi) \leq a_{\alpha,\beta}(1+|\beta|)^{-|\alpha|} \tag{21}$$

$\forall \alpha, \beta$ are multi indeces. Then we consider dyadic decomposition, the is a function ϕ satisfied the condition in 19, define $a_i(x,\xi) = \phi(2^{-i}x)a(x)$. then $suppa_i(x,\xi)$ cpt, $\|a_i\| < \infty$. So $a_i \in L^p(\mathbb{R}^n)$, we have,

$$
\begin{aligned}
\|T_{a_i}f\|_p^p &= \int_{\mathbb{R}^d} |K_i * f(x)|^p dx \\
&= \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} K_i(x-y)f(y)dy|^p dydx \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K_i(x-y)f(y)|^p dydx \\
&= \|K_i\|_p^p \|f\|_p^p
\end{aligned}
$$

$\|K_i\|_p$ have good decay estimate, thanks to $u_i = \phi(2^{-i}x)a(x) \in S^0$, this estimate is deduce morally along the same ingredient of "station phase", it is come from a argument combine "counting point" argument and a rescaling argument. So,

$$
\begin{aligned}
\|Tf\|_p &= \|\sum T_i f\|_p \\
&\leq (\sum \|K_i\|_p^p)\|f\|_p
\end{aligned}
$$

But we have $\sum \|K_i\|_p^p \leq \infty$, ending the proof. □

Remark 13 *this method also make sense of restrict the condition to be:*

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha,\beta}(1+|\xi|)^{-|\alpha|}, \forall |\alpha| \leq d+2. \tag{22}$$

Where d is the dimension of the space, and we could change 2 to $1+\epsilon$.

Remark 14

$$\widehat{\frac{\partial^2 u}{\partial x_i \partial x_j}}|\xi| = \frac{\xi_i \xi_j}{|\xi|^2}\widehat{\Delta}u(\xi), m(\xi) = \frac{\xi_i\xi_j}{|\xi|^2} \tag{23}$$

is a counter example for $p = 1, \infty$.

Remark 15 *The key point is the estimate*

$$\int |\int e^{2\pi i \xi x}\phi(2^{-j}\xi)m(\xi)d\xi|^p dx \tag{24}$$

Correlation of taylor expension and wavelet expension.

This is also crutial for the theory of station phase.

4 Calculus of symbols

This calculus of symbols would imply there is some structure on this set.

Theorem 16 *Suppose a, b are symbols belonging to S^{m_1} and S^{m_2} respectively. Then there is a symbol c in $S^{m_1+m_2}$ so that:*

$$Tc = T_a \circ T_b$$

Moreover,

$$c \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha} (\partial_\xi^\alpha a)(\partial_x^\alpha b). \tag{25}$$

in the sense that,

$$c - \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha b \in S^{m_1+m_2-N} \tag{26}$$

For all $N > 0$.

The following "proof" is not rigorous, we just calculate it formally, we could believe it is true rigorously, by some approximation process. PROOF: We assume a, b have compact support so that our manipulations are justified. We use the alternate formula 15 to write,

$$(T_a f)(y) = \int b(y, \xi) e^{2\pi i \xi (y-z)} f(z) dz d\xi \tag{27}$$

Then we apply T_a, again in the form 15, but here with the variable η replacing in the integration. The result is,

$$T_a(T_b f)(x) = \int a(x, \eta) b(y, \xi) e^{2\pi i \eta (x-y)} e^{2\pi i \xi (y-z)} f(z) dz d\xi dy d\eta. \tag{28}$$

This calculate is easy to derive, but the following is more tricky. Now $e^{2\pi i \eta (x-y)} \cdot e^{2\pi i \xi (y-z)} = e^{2\pi i (x-y)(\eta - \xi)} \cdot e^{2\pi i (x-z)\xi}$, so

$$T_a(T_b f)(x) = \int c(x, \xi) e^{2\pi i (x-z)\xi} f(z) dz d\xi \tag{29}$$

with

$$c(x, \xi) = \int a(x, \eta) b(y, \xi) e^{2\pi i (x-y)(\eta - \xi)} dy d\eta \tag{30}$$

we can also carry out the integration in the y-variable. This leads to the corresponding Fourier transform of b in that variable, and allows us to rewrite 30 as,

$$c(x, \xi) = \int a(x, \xi + \eta) \hat{b}(\eta, \xi) e^{2\pi i x \eta} d\eta. \tag{31}$$

With this form in hand, use taylor expense to the symbol $a(x,\xi+\eta)$, i.e.

$$a(x,\xi+\eta)=\sum_{|\alpha|<N}\partial_\xi^\alpha a(x,\xi)\eta^\alpha+R_N(x,\xi,\eta) \tag{32}$$

with a suitable error term R_N, due to

$$\frac{1}{\alpha!}\int\partial_\xi^\alpha a(x,\xi)\hat{\eta}(\eta,\xi)e^{2\pi ix\eta}d\eta=\frac{(2\pi i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi)). \tag{33}$$

we only need to proof $R_N\in S^{m_1+m_2-N}$ and it is definitely the case, we get the theorem. $\square$

Remark 17 *We need replace a,b with a_ϵ,b_ϵ, where*

$$a_\epsilon(x,\xi)=a(x,\xi)\cdot\gamma(\epsilon,\epsilon\xi),b_\epsilon(x,\xi)=b(x,\xi)\cdot\gamma(\epsilon,\epsilon\xi). \tag{34}$$

we note that a_ϵ,b_ϵ satisfy the same differential inequalities that a and b do, uniformly in $\epsilon,0<\epsilon\leq 1$ .passage to the limit as $\epsilon\to 0$ will then give us our desired result.

5 Estimate in L^p , Sobolev, and Lipchitz space

We now take up the regularity properties of our pesudo-differential operator as expressed in terms of the standard function spaces, we begin with the L^p boundedness of an operator of order 0.

5.1 L^p estimate

Suppose a belongs to the symbol class S^0. Then, we can express $T=T_a$ as

$$(Tf)(x)=\int K(x,y)f(y)dy=\int K(x,x-y)f(y)dy \tag{35}$$

due to $a\in S^0$, we know, with some approximation argument and first do it with a cutoff symbol of a, i.e. a_ϵ, that,

$$|K(x,y)|\leq A|x-y|^{-n} \tag{36}$$

So that the integral coverage whenever $f\in S$ and x is away from the support of f. Since we know that T is bounded on $L^2(\mathbb{R})$, this representation extends to all $f\in L^2(\mathbb{R})$ for almost every $x\notin suppf$. More generally, we have,

$$|\partial_x^\alpha\partial_y^\beta K(x,y)|\leq A_{\alpha,\beta}|x-y|^{-n-|\alpha|-|\beta|} \tag{37}$$

hence K satisfies,

$$\int_{|x-y|\geq 2\delta} |K(x,y)-K(x,\bar{y})|dx \leq A,\ if\ |y-\bar{y}|\leq \delta, all\ \ \delta>0. \tag{38}$$

Use the general singular integral theory we get the following L^p estimate.

Theorem 18 *Suppose T_a is the pseudo-differential operator corresponding to a symbol a in S^0, then T_a extends to a bounded operator on $L^p(\mathbb{R}^n)$ to itself, for $1<p<\infty$.*

5.2 Sobolev spaces

We first recall the definition of the Sobolev spaces W_k^p, where k is a positive integer. A function f belongs to $W_k^p(\mathbb{R}^n)$ if $f\in L^p(\mathbb{R}^n)$ and the partial derivatives $\partial_x^\alpha f$, taken in the sense of distribution, belong to $L^p(\mathbb{R}^n)$, whenever $0\leq|\alpha|\leq k$. The norm in W_k^p is given by,

$$\|f\|_{W_k^p} = \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_{L^p} \tag{39}$$

the following result is the directly corollary of 18.

Theorem 19 *Suppose T_a is a pseudo-differential operator whose symbol a belongs to S^m. If m is an integer and $k\geq m$, then T_a is a bounded mapping from W_k^p to W_{k-m}^p, whenever $1<p<\infty$.*

Remark 20 *This theorem remain valid for arbitrary real k,m.*

5.3 Lipschitz spaces

Theorem 21 *Suppose a is a symbol in S^m. Then the operator T_a is a bounded mapping from Λ_γ to $\Lambda_{\gamma-m}$, whenever $\gamma>m$.*

Lemma 22 *Suppose the symbol a belongs to S^m, and define $T_{a_j}=T_a\Delta_j$. Then, as operator from $L^\infty(\mathbb{R}^n)$ to itself, the T_{a_j} have norms that satisfy*

$$\|T_{a_j}\| \leq A2^{jm} \tag{40}$$

We shall now point out a very simple but useful alternative characterization of Λ_γ. This is in terms of approximation by smooth functions; it is also closely connected with the definition of Λ_α space as intermediate spaces, using the "real" method of interpolation.

Corollary 23 *A function f belongs to Λ_γ if and only if there is a decomposition,*

$$f = \sum_{j=0}^{\infty} f_j \tag{41}$$

with $\|\partial_x^\alpha f_j\|_{L^infty} \leq A 2^{-j\gamma} \cdot 2^{j|\alpha|}$, for all $0 \leq |\alpha| \leq l$, where l is the smallest integer $> \gamma$.

When $f \in \Lambda_{/\gamma}$, the argument prove 22, with $T_a = I$, $f_j = F_j = \Delta_j(f)$, gives the required estimate for the f_j.

A second consequence of 21 is the following:

Corollary 24 *The operator $(I - \Delta)^{\frac{m}{2}}$ gives an isomorphism from Λ_γ to $\Lambda_{\gamma - m}$, whenever $\gamma > m$.*

This is clear because $(I - \Delta)^{\frac{m}{2}}$ is continuous from Λ_γ to $\Lambda_{\gamma-m}$, and its inverse, $(I - \Delta)^{\frac{-m}{2}}$, is continuous from $\Lambda_{\gamma-m}$ to Λ_γ.