Calderon-Zegmund decomposition and singular integral

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1 Calderon-Zygmund decomposition

The Calderon-Zygmund decomposition is a key step in the real variable analysis of singular integrals. The idea behind this decomposition is that it is often useful to split an arbitrary integrable function into its "small" and "large" parts, and then use different technique to analyze each part.

The scheme is roughly as follows. Given a unction f and an altitude α, we write $f = g + b$, where $|g|$ is point wise bounded by a constant multiple α. While b is large, it does enjoy two redeeming features: it is supported in a set of reasonable small measure, and its mean value is zero on each of the ball that constitute its support. To obtain the decomposition $f = g + b$, one might be tempted to "cut" f at the height α; however, this is not what works. Instead, one bases the composition on the set where the maximal function of $f$has height α.

Theorem 1 *(Calderon-Zygmund decomposition)*
Suppose we are given a function $f \in L^1$ and a positive number α, with $\alpha > \frac{1}{\mu(R^n)} \int_{R^n} |f| d\mu$. Then there exists a decomposition of f, $f = g + b$, with latex$b = \sum_k b_k$, and a sequences of balls $\{B_k^*\}$, so that,

1. *$|g(x)| \leq c\alpha$, for a.e. x.*
2. *Each latexb_k is supported in B_k^*,$\int |b_k(x)| d\mu(x) \leq c\alpha\mu(B_k^*)$, and $\int b_k(x) d\mu(x) = 0$.*
3. *$\sum_k \mu(B_k^*) \leq \frac{c}{\alpha} \int |f(x)| d\mu(x)$.*

Before proof this theorem, I explain the geometric intuition why this theorem could be true first. Merely speaking, this is just base on cut off the function into two part, the part with high altitude and the part with low altitude and extension the part with high altitude to make the extension one satisfied the condition 2 and 3.

PROOF:[rough] In fact this decomposition have a good geometric explain, we just divide the part $\{x : |f(x)| > \alpha\}$ and extension it carefully to make they behaviour like several balls, to satisfied the special condition on this part. □

Remark 2 *Remark 1:A Calderon-Zygmund decomposition for L^p function was done in Charlie Fefferman's thesis; see Section II of ams.org/mathscinet-getitem?mr=257819*

One can also find this in Loukas Grafakos's Classical Fourier Analysis Classical Fourier Analysis page 303 exercise 4.3.8. The question is broken up into parts that should be easy to handle.

Several people have considered with this question. An excellent paper that comes to mind is Anthony Carbery's Variants of the Calderon–Zygmund theory for L^p-spaces which appeared in Revista Matematica Iberoamericana, Volume 2, Number 4 in 1986. There are also several useful references that appear in Carbery's paper.

Remark 3 *We could also consider a variant of Calderon-Zygmund decomposition, such as equipped with a nontrivial weight function w or find some different way to decomposition for some special purpose.*

Remark 4 *Consider suitable decomposition of the physics space or even both the physics space and fractional space try to gain some reasonable estimate is a fundamental philosophy in harmonic analysis, beside the Calderon-Zegmund decomposition,*

Whitney decomposition. Which is important trick in the proof of fefferman-stein restriction theorem and differential topology.

Wave packet decomposition.The wave packet decomposition. This decomposition underlies the proof of Carleson's theorem (this is more explicit in Fefferman's proof than Carelson's original proof), Lacey and Thiele's proof of the boundedness of the bilinear Hilbert transform, as well as a host of follow-up work in multilinear harmonic analysis. The idea of the wave packet decomposition is to decompose a function/operator in terms of an overdetermined basis. This allows one to preserve symmetries (such as modulation symmetries) that aren't preserved by a classical Calderon-Zygmund decomposition (which endows thefrequency with a distinguished role). One might consider using a wave packet decomposition if is working with an operator that has a modulation symmetry. This is discussed in more detailed in Tao's blog post on the trilinear Hilbert transform.

Polynomial decomposition.The application of polynomial decomposition to harmonic analysis is more recent, and its full potential still seems unclear. Applications include

Dvir's proof of the finite field Kakeya conjecture, Guth's proof of the endpoint multilinear Kakeya conjecture (and, indirectly, the Bourgain-Guth restriction theorems), Katz and Guth's proof of the joints problem and Erdos distance problem, among many other results. Generally, the idea behind the polynomial decomposition is to partition a subset of a vector space over a field into a finite number of cells each of which contains roughly the same fraction of the original set. One further wishes that no low degree algebraic variety can intersect too many of the cells. In Euclidean space, the polynomial ham sandwich decomposition does exactly this. This allows one to, for instance, control linear (or, more generally, 'low algebraic degree') interactions between points in distinct cells. This has so far proven the most useful in incidence-type problems, but many problems in harmonic analysis, thanks to the translation symmetry of the Fourier transform, are inextricably linked with such incidence-type problems. See (again) Tao's survey of this topic for a more detailed account.

2 Singular integrals

Have the Calderon-Zegmund decomposition in hand, now we proof a conditional one bounded result for singular integrals.

The singular integral one is interested in are operator T, expressible in the form

$$(Tf)(x) = \int_{R^n} K(x,y)f(y)d\mu(y) \tag{1}$$

Where the kernel K is singular near $x = y$, and so the expression is meaningful only if K is treated as a distribution or in some limiting sense. Now the particular regularization of $(Tf)(x)$ may be appropriate depends much on the context, and a complete treatment of the issues thereby raised take us quite far afield.

Let us limit ourselves to two closely related ways of dealing with the questions concerning the definability of the operator. One is to prove estimates for the (dense) subspace where the operator is initially defined. The other is to regularize the given operators by replacing it with a suitable family, and to prove the uniformly estimates for this family. This idea is similar occurring in spectral geometry when we wish to investigate the spectrum of some operator we try to consider some deformation, so deduce to control the spectrum of a seres of paramatrix, for example, consider the wave kernel or heat kernel rather than the passion kernel itself. Common to both methods is a priori approach: We assume some additional properties of the kernel, but then prove estimates that are independent of these "regularity" properties.

We now carry out the first approach in detail. There will be two kinds of assumptions made about the operator. The first is quantitative: we assume that we are given a bound A, so that the operator T is defined and bounded on L^q with norm A; that is,

$$\|T(f)\|_q \leq A\|f\|_q, \forall f, f \in L^q \tag{2}$$

Moreover, we assume that there is associated to T a measurable function K (that plays the role of its kernel), so that for the same constant A and some constant $c > 1$,

$$\int_{R^n - B(y,c\delta)} |K(x,y) - K(x,\bar{y})| d\mu(x) \leq A, \forall \bar{y} \in B(y,\delta) \tag{3}$$

for all $y \in R^n, \delta > 0$.

The further regularity assumption on the kernel K is that for each f in L^q that has compact surppot, the integral coverages absolutely for almost all x in the complement of the support of f, and that equality holds for these x.

Theorem 5 *(Bounded of singular integral with condition)*

Under the condition 1 and 3 made above on K, the operator T is bounded in L^p norm on $L^p \cap L^q$, when $1 < p < q$. More precisely,

$$\|T(f)\|_p \leq A_p\|f\|_p$$

For $f \in L^p \cap L^q$ with $1 < p < q$, where the bound A_p depends only on the constant A appearing in 1 and 3 and on p, but not on the assumed regularity of K, or on f.

PROOF:

Now let us begin to prove the conditional theorem. The key point is to use the potential of T has been a bounded operator from $L^q \to L^q$. Said, it already assumed $\exists A > 0$ such that $\forall f \in L^q$ we have $\|T(f)\|_q \leq A\|f\|_q$. Now let us look at the singular integral expression:

$$(Tf)(x) = \int_{R^n} K(x,y)f(y)d\mu(y). \tag{4}$$

The key point is to proof the mapping $f \to T(f)$ is a weak-type $1-1$; that is,

$$\mu\{x : |Tf(x)| > \alpha\} \leq \frac{A'}{\alpha}\int |f| d\mu. \tag{5}$$

At once we establish 5, then the theorem followed by interpolation. Now we use theorem 1 on f get $f = g + b$, thanks to the triangle inequality and something similar we have $g, b \in L^q$, in fact $R^n = A \amalg B, B \cup_k B_k$, $g = \chi_A g + \chi_B g, b = \chi_A b + \chi_B b$, by triangle inequality and $f = g + b$, to proof $g, b \in L^q$, we only need to proof $\chi_A g, \chi_B g, \chi_A b, \chi_B b \in L^q$, but this is easy to proof.

Now we know the L^q bounded of g, b, we divide the difficult of establish the weak 1-1 bound of f into the difficult of establish the weak 1-1 bound for g and b. i.e.

$$\mu\{x : |Tf(x)| > \alpha\} \leq \mu\{x : |Tg(x)| > \alpha\} + \mu\{x : |Tb(x)| > \alpha\} \tag{6}$$

For g, if this weak 1-1 bound is not true, we have,

$$\mu\{x : |Tg(x)| > \alpha\} \geq \frac{A'}{\alpha} \int |g| d\mu \tag{7}$$

thanks to the trivial estimate $\|g\|_q \leq c\alpha^{q-1}\|g\|_1$. combine this two estimate we have:

$$c\alpha^{q-1}\|g\|_1 \geq \|g\|_q^q \geq c\|Tg\|_q^q \geq A'\alpha^{q-1}\|g\|_1 \tag{8}$$

The first estimate is true on A due to $|g| \leq \alpha, a.e.x \in R^n$. But compare the left and the right of 8 lead a contradiction, so 7 follows.

For b, the thing is more complicated and in fact really involve the structure of the convolution type of the singular integral. The key point is controlling near the diagonal of $K(x, y)$. we warm up with a more refine decomposition $b = \sum b_k$, $\forall k, b_k = b \cdot \chi_{B_k}$. For a large constant $c >> 1$ choose later define $B_k^* = cB_k$. We know $b \in L^q$, but the really difficult thing occur in the how to combine the following 5 condition to lead a contradiction:

1. $\|Tb\|_q \leq \|b\|_q$.
2. property come from the Calderon-Zegmund decomposition, $\int_{B_k} \|b\| \leq c\alpha\mu(B_k), \forall k$ and $\int_{B_k} b = 0$.
3. Hormander condition 3 , $\int_{R^n - B(y,c\delta)} |K(x, y) - K(x, \bar{y})| d\mu(x) \leq A, \forall \bar{y} \in B(y, \delta)$
4. the reverse of weak 1-1 of b, $\mu\{x : b(x) > \alpha\} > \frac{A'}{\alpha}\|b\|_1$.
5. the structure $Tb(x) = \int_{R^n} K(x, y)b(y)dy$

The first step is to break b into b_k, and reduce the case of several balls to the case of only one ball, this could be done by triangle inequality or more may be we could do it derectly, but any way it is not difficult.

Then the thing become intersting, we focus on b_1, divide $Tb_1 = T\chi_{B_1}b_1 + T\chi_{\mathbb{R}^n - B_1}b$. thanks to the hormander condition 3 we have good control on $T\chi_{\mathbb{R}^n - B_1^*}$, in fact we can proof a weak 1-1 bound on it,

$$\mu\{x : |T_{\mathbb{R}^n - B_1^*}b_1| > \alpha\} < \frac{A'}{\alpha}\|b_1\|_1 \tag{9}$$

$$
\begin{aligned}
T_{\mathbb{R}^n - B_1^*} b_1(x) &= \int_{\mathbb{R}^n - B_1^*} K(x,y) b_1(y) dy \\
&= \int_{\mathbb{R}^n - B_1^*} [K(x,y) - K(x,\bar{y})] b_1(y) dy + \int_{\mathbb{R}^n - B_1^*} K(x,\bar{y}) b_1(y) dy \\
&\leq \int_{\mathbb{R}^n - B_1^*} A b_1(y) dy + \int_{\mathbb{R}^n - B_1^*} K(x,\bar{y}) b_1(y) dy
\end{aligned}
$$

So we conclude,

$$
\begin{aligned}
\|T_{\mathbb{R}^n - B_1^*} b_1\|_1 &= \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n - B_1^*} K(x,y) b_1(y) dy| dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n - B_1^*} |[K(x,y) - K(x,\bar{y})] b_1(y) dy| dx + \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n - B_1^*} K(x,\bar{y}) b_1(y) dy| dx \\
&\leq A \int_{\mathbb{R}^n} b_1(y) dy + \int_{\mathbb{R}^n - B_1^*} K(x,\bar{y}) b_1(y) dy = A \int_{\mathbb{R}^n} b_1(y) dy
\end{aligned}
$$

The last equality used the condition $\int b_1 = 0$.

□