Brunn-Minkowski inequality

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Abstract

In this short note, I posed a conjecture on Brunn-Minkwoski inequality and explain why we could be interested in this inequality, what is it meaning for further developing of some fully nonlinear elliptic equation come from geometry. The main part of the note devoted to discuss several different proof of classical Brunn-Minkowski inequality.

Contents

1 Introduction

I believe, every type of Brunn-Minkowski inequality, type of Brunn-Minkowski inequality is in some special sense and will be explained later, will be crucial with a corresponding regularity result of a fully nonlinear elliptic equation which could be realizable by geometric way which will also explained in further note.

So the key point is that Brunn-Minkowski inequality is crucial and have potential application, I posed a problem there and then consider the classical Brunn-Minkowski inequality, we give several proof of the classical Brunn-Minkowski inequality, everyone could help us to have a more refine understanding of the original difficulty with different angle.

Theorem 1 (conjecture) *We have a map*

$$f : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \tag{1}$$

$$(x, y) \longrightarrow f(x, y)$$

We are willing to called the function f as the hamiltonian function. then we could consider the hamiltonian flow of the function f, but this could only true for a even dimension manifold to make there exists f that df is a non-degenerate closed $2-$form.

Anyway we consider the level set of f, we get a foliation i.e $\mathbb{R}^d = \{\amalg_{p\in\mathbb{R}} f^{-1}(p)\}$. we consider the gradient flow with f, called the gradient flow begin with $q \in \mathbb{R}$ as $\{\phi_q(t)\}_{t\in\mathbb{R}}$. And we wish the gradient flow have a addition structure on itself then we could consider what is the Brunn-Minkowski inequality in this setting, the condition is a group structure on the space of level set $L_f = \{\amalg_{p\in\mathbb{R}} f^{-1}(p)\}$, i.e.

$$\forall t_1, t_2 \in \mathbb{R}, \forall q \in R^d, \phi_q(t_1 + t_2) = \phi_q(t_1) \circ \phi_q(t_2) \tag{2}$$

Remark 2 *take $f(x, y) = x + y$ in 1, this conjecture reduce to the toy model, i.e. classical Brunn-Minkowski inequality.*

Remark 3 *We could generate the problem to the problem which is charged by several energy function $f_1, ..., f_k$, if the induced gradient flow is amenable, then this is somewhat similar with the one dimension case, I wish if we could do something for the single function f_1, then we can say something for the several functions involved case.*

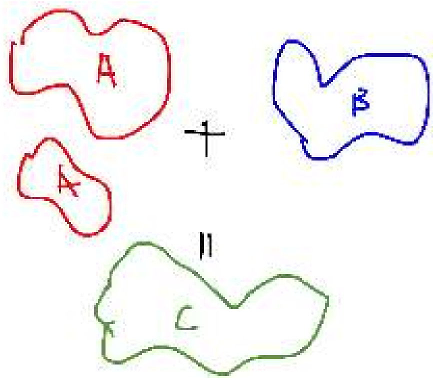
Remark 4 *This could also generate to amenable group action case and quantization of it.*

Meaning, the cohomology induce by a hamiltonian system on some special foliation on fiber of geometric bundle. This type of result could help to establish the vanish of the cohomology, the get the existence theory and regularity result for corresponding elliptic nonlinear differential equation. And solve the original problem I consider.

Now we given the statement of Brunn-Minkowski inequality.

Theorem 5 (brunn minkowski inequality) *For A, B measurale set in $\mathbb{R}^d$. we have following,*

$$\mu^{\frac{1}{d}}(A+B) \geq \mu^{\frac{1}{d}}\mu(A) + \mu^{\frac{1}{d}}\mu(B) \tag{3}$$



of sets.png

The general spproach of Brunn-Minkwoski inequality is following,

1. divide the measurable set A, B into small cubes.
2. Shinking trick, transform the set into convex one.

for the first one, we have the following lemma,

Lemma 6 $\forall 0 < \lambda < 1$, $\exists \epsilon > 0$ A_ϵ *measurable set,* $A_\epsilon \subset A$, $\mu(A - A_\epsilon) < \epsilon$, *and* $A = (A - A_\epsilon) \amalg \cup_{i \in I}(c_i \cap A_\epsilon)$, *and*

$$\frac{\mu(c_i \cap A_\epsilon)}{\mu(c_i)} > \lambda, \forall i \in I \tag{4}$$

PROOF: The proof of the lemma is a easy corollary of the construction of Lesbegue(or Borel) measurable σ algebra. $\square$

Remark 7 *The existence of the property given in the lemma is not the key point, the key point is* $d(A_\epsilon, A) \to 0, as \ \ \epsilon \to 0$.

Has this two simplify in hand, we could give several approach to proof the inequality and these proof carry information more than just a proof, they carry some information with the structure of space $(\{0,1\}_{\mathbb{R}^d}, \mu^{\frac{1}{d}}, +)$.

2 A proof with discretization

There is a lots of ways to attack the Brunn-Minkowski inequality, the most natural one is discretization. But unfortunately there is some technique obstacle for proof or even state the discretization version of "Brunn-Minkowski" inequality.
The "boundary" and "area" should not compatible.

$$\mu_{d-1}(\partial E) << \mu_d(\mu(E)) \tag{5}$$

And we need use the fact,

$$A_\epsilon \stackrel{G-H\ sence}{\longrightarrow} A \tag{6}$$

Now we just state what we expect it should transform in, because we have a fully understanding with the discretization model, there is a result named Cauchy-Daveport inequality.

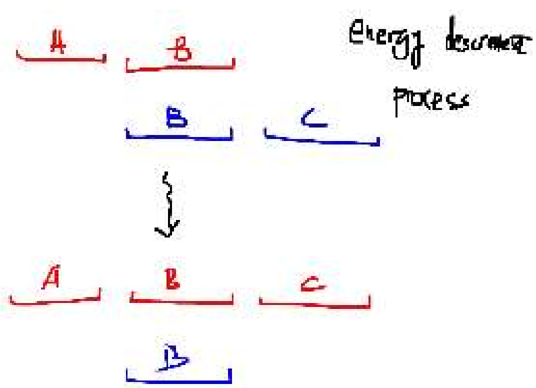
Theorem 8 (cauchy-daveport inequality) *There are two case, one in $\mathbb{Z}$, one in finite field $\mathbb{Z}_p$.*

1. *$\mathbb{Z}$ case, $\forall A, B \subset \mathbb{Z}$ are finite set,we have,*

$$|A + B| \geq |A| + |B| - 1. \tag{7}$$

2. *$\mathbb{Z}_p$ case, $\forall A, B \subset \mathbb{Z}$ are finite set,we have,*

$$|A + B| \geq \max\{|A| + |B| - 1.p\}. \tag{8}$$



decrement.png

Proof: for the $\mathbb{Z}$ case, the story is more or less trivial, just do to a observation, if $A = \{a_i, a_1 < a_2 < ... < a_n\}, B = \{b_i, b_1 < b_2 < ... < b_m\}$, then

$$a_1 + b_1 < \min\{a_1 + b_2 + a_2 + b_1\} < .... < a_n + a_m$$

There exists a strictly increasing chain of length at least $n + m - 1$.

For the $\mathbb{Z}_p$ case, following is a graph to explain what happen, basically we define a operation on tuples, i.e. $T : (A, B) \to (T(A), T(B))$, and make the additive energy $E_{A,B} := |A+B|$ decreasing. after induction with this transform and the transform from a tuple to the minimum additive energy by translation, the additive energy decreasing and decreasing then arrive the global minimum. But it is easy to conclude in this case one of A, B become null set and then the inequality 8 follows. $\square$

But when we discrete the Brunn-Minkowski inequality, we expect a high dimension generation of the inequality 8. Naively we wish,

Theorem 9 (naive generation of cauchy-daveport inequality) *For $d \in \mathbb{N}$, and $\forall A, B \subset \mathbb{Z}_d$ are finite sets,*

$$|A+B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \tag{9}$$

But this is not the case, there is a counterexample for 9. We could construct some A, B such that $|A+B| \sim |A| + |B|$, consider they be very thin line.

So why we are in this worse situation? because we lose the information of $A_\epsilon \overset{G-H}{\longrightarrow} A$, $B_\epsilon \overset{G-H}{\longrightarrow} B$. So they have the trend tending to make the "boundary" campatible with "area". Two thin line in the same direction is exactly the worst case, which is just a equal condition of 1-dimension case.

One natural way to except the situation is to bounded the "isperimetric constant", to assume A, B varies in a subset of measurable set, with addition condition that $\frac{\mu_{d-1}(\partial A)}{\mu_d(A)}$ is bounded by some constant. But this is also not the suitable set for our inequality, I explain how to capture the information of the G-H coverage.

Now assume A, B are convex bounded set, and we take a global orthogonal basis in $\mathbb{R}^d$. named $(e_1, ..., e_d)$. We give the definition of $\epsilon-$discretization of A, named A_ϵ.

Definition 10 ($\epsilon-$ discretization) *The construction of A_ϵ from A is following:*

1. *divide A into $\amalg_{i\in I} c_i \cap A$, c_i is the ϵ cubes.*
2. *use c_i or $\emptyset$ instead of $c_i \cap A$ depending on iff $\frac{\mu(A\cap c_i)}{\mu(c_i)} > \lambda$, where $\lambda < 1$ is a given number only rely on ϵ. i.e.*

$$A \cap c_i \to D_\epsilon(A \cap c_i) \tag{10}$$

3. *glue them, define $A_\epsilon := \amalg_{i\in I} D_\epsilon(c_i \cap A)$.*

Now we describe the condition of $A_\epsilon \overset{G-H}{\to} A$ rigorous meaning campatible with $\epsilon-$discretization.

Under the basis, there is a coordinate we could know iff $c_i = c(\lambda_1, ..., \lambda_d)$ is the cube center at $(\epsilon\lambda_1, ..., \epsilon\lambda_d)$ if it is in A_ϵ. Due to A is convex, ∂A is lipchitz. So you will have some locolization property, said, at every fix discretization scale ϵ, the position of $(\lambda_1\epsilon, ..., \lambda_d\epsilon)$ is morally known so the number of cubes in A_ϵ in the one dimensional affine space $\Omega^\epsilon_{a_1,...,a_{k-1},a_{k+1},...,a_n}$ which is the subspace of $\mathbb{Z}^d$

the number $\Omega^{\epsilon}_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n}$ is asymptopic to the $1-$dimensional hausdorff measure of $A \cap \Omega_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n}$. So at least,

$$|\Omega^{\epsilon}_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n}| = O((\frac{1}{\epsilon})) \tag{11}$$

Property 11 is crucial, which mean A_ϵ is really a n-dimensional space and automatically we have the bounded on isoperimetric constant $\frac{\mu_{d-1}(\partial A)}{\mu_d(A)}$.

Now we can look at every A_ϵ and take limit $\epsilon \to \infty$. In fact we a in the situation with Accumulation of wood to make the product have smallest volume. Not to optimized the tuples (A, B) but fix one of it, said B, optimized the other one, said A. This is the key point of proof, a little bit different from the argument of $1-$dimensional 8 where we optimized the tuple.

Key point:

1. we can ignore "small core".
2. This inequality is said, due to $A + B = (\frac{A+B}{2}) + \frac{A+B}{2}$, the convex of the functional $\mu^{\frac{1}{d}}$ on convex set.

The way of discretization could not handle the problem but definitely said that the difficulty occur with the shape of boundaries $\partial A, \partial B$.

3 A proof with "central of mass" and Minkowski functional

Definition 11 (Central of mass) *The central of mass p of measurable set A, if exist, satisfied, $\forall e \in \S^d$, there is a subspace L_e with codimension 1 divide A into two connected part A_{1,L_e}, A_{2,L_e} such that*

$$\mu(A_{1,L_e}) = \mu(A_{2,L_e}) \tag{12}$$

then $p \in l_e$.

Remark 12 *For a measurable set A , if central of mass p exists, then there exist only one. This is a easy observation do to the definition of p, i.e. the intersection of suitable affine subspace in every direction.*

Theorem 13 (existence of central of mass)

PROOF: It is easy to attain p by take $n+1$ different directions in S^d, then easy to proof every line L_e across it be definition of L_e. $\square$

Definition 14 (Minkowski functional) *for a measurable set $X \subset \mathbb{R}^d$ and a point p, define $M_{X,p}$ on S^{d-1}, such that*

$$M_X(e) = \sup_{\lambda > 0, \lambda e + p \in X} \lambda \tag{13}$$

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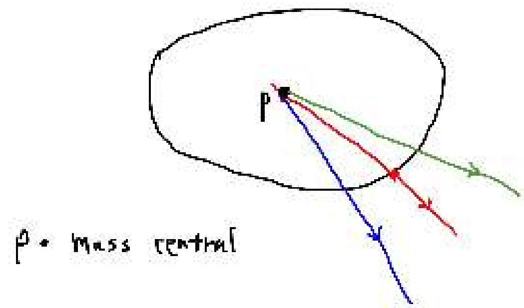

Remark 15 *If A is convex, then M_A is a convex function on S, so it is lipchitz.*

we have following formula for the measure of A.

Theorem 16

$$\mu(A) = \frac{1}{\mu(S^{d-1})} \int_{S^{d-1}} M_A(e) de \tag{14}$$

Proof: trivial. □ Now the task reduce fixing B and $\mu(A)$ to optimized A make $\mu(A+B)$ small. It is the same as make $\frac{1}{\mu(S^{d-1})} \int_{S^{d-1}} M_{A+B}(e) de$ small when fix $\frac{1}{\mu(S^{d-1})} \int_{S^{d-1}} M_A(e) de$ and M_B. Due to

$$M_{A+B}(x) = \sup_{x_1, x_2 \in S^{d-1}} M_A(x_1) + M_B(x_2) \tag{15}$$

This lead to the whole story, given a proof of 3.

4 A proof with multi-scale analysis

This approach is a nonstandard one, due to I believe the renormlization or continue fractional or multilinear estimate is everywhere. We first play with a toy model, the rectangle.

Theorem 17 *Brunn-Minkowski inequality is right for A, B are rectangles.*

Proof:

$$(\Pi(a_1 + b_i))^{\frac{1}{d}} \geq (\Pi(a_1))^{\frac{1}{d}} + (\Pi(a_1 + b_i))^{\frac{1}{d}}$$

$$\Leftrightarrow 1 \geq [\frac{\Pi a_i}{\Pi(a_i + b_i)}]^{\frac{1}{d}} + [\frac{\Pi a_i}{\Pi(a_i + b_i)}]^{\frac{1}{d}} \tag{16}$$

In fact,

$$\begin{aligned} 16 RHS &\overset{A-G}{\leq} \frac{1}{d} \sum_i \frac{a_i}{a_i + b_i} + \frac{1}{d} \sum_i \frac{b_i}{a_i + b_i} \\ &= 1. \end{aligned}$$

□ The story is following,

5 connection of Brunn-Minkowski inequality and Sobolev inequality, the firth proof

We begin with a calculate based on intuition and it is not rigorous.

$$
\begin{aligned}
\mu(A+B)^{\frac{1}{d}} &= [\int_{\mathbb{R}}(\int_{\mathbb{R}^{d-1}} \chi_{A+B}(\xi_1,...,\xi_{n-1})d\xi_1...d\xi_{n-1})d\xi_n]^{\frac{1}{d}} \\
&\sim \frac{1}{\mu(A_n)}\int_{\mathbb{R}}(\int_{\mathbb{R}^{d-1}} \chi_{A+B}(\xi_1,...,\xi_{n-1})d\xi_1...d\xi_{n-1})^{\frac{1}{d-1}}d\xi_n \\
&\overset{induction\ on\ d}{\geq} \frac{1}{\mu(A_n)}(\mu^{\frac{1}{d-1}}(A(\xi_n))+\mu^{\frac{1}{d-1}}(B(\xi_n)))d\xi \\
&\overset{A-G\ inequality}{\geq} \mu^{\frac{1}{d}}(A)+\mu^{\frac{1}{d}}(B).
\end{aligned}
$$

The second line is due to I believe there $\exists\ A, B$ such that it is a equality, by the equal condition of Minkowski inequality, in fact this is morally inverse of Minkowski inequality. The second reason in general case why the second inequality is true is due to a rescaling argument, change $(\xi_1,...,\xi_n) \to (\lambda\xi_1,...,\lambda\xi_n), \forall\lambda \in \mathbb{R}$, by the rescaling argument we conclude if there is a such inequality, the index of it must be the case.