Further example of singular integral

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1 Introduction

We have talked about a very basic result in singular integral, i.e. if we have an additional condition, i.e. $q - q$ bounded condition, then by interpolation theorem we only need to establish the weak $1 - 1$ bound then we establish the $p - p$ bound of T, $\forall 1 < p < q$.

The category of of singular integral is very general, in fact the singular integral we interested in always equipped more special structure. We discuss following 3 types result which world be the central role in this further series note.

1. Approximation of the identity.
2. Singular integral with L^2 bounded translation invariant operator.
3. Maximal function, singular integral, and square functions.

The underlying object we consider in both the three case is some special singular integral, in the first case, it looks like a $T = sup_{t>0}\Phi_t * f$, this, among the other thing, has a close relationship with the maximal operator Mf. This is discussed in 2. For the singular integral with L^2 bound, the Fourier transform or its discretization version, Fourier series is natural involved. And there is a "representation theorem" similar to the sprite of Reisz representation theorem, said, roughly speaking, if we consider the L^2 bound operator adding the condition of transform invariant, then it is really coinside with the case of our image, the operator must behaviour as a Fourier multiple. This is the contant of famous Mikhlin multiplier theorem, and we discuss some technique difficulty in the process of establishing such a theorem, this is the contact of 3. At last we discuss some deep relationship between three basic underlying intution and objects in harmonica analysis, the Maximal function, singular integral, and square functions. They could all be understanding as tools to understanding the variant complicated emerging in singular integral. But there is definitely some common points. This is the theme of 4. Of course there are some further topic which are also interesting, but I do not want to discuss them here, maybe somewhere else.

2 Approximation of the identity

First topic, we discuss the approximation of the identity, this play a central role in understanding solution of PDE, why, I think a key point is this tools carry a lots of information about the scaling of the space, as it well known, analysis could roughly divide into two parts, "hard analysis" and "soft analysis", approximation of the identity supply a way to transform a result form "hard analysis" side to "soft analysis" side and reverse. And when it shows its whole power always along with the involving of following Dominate convergence theorem:

Theorem 1 *(DCT)*

Let $\{f_n\}_{n=1}^{\infty}$ *be a series of function on measure space* (X, Σ, μ)*, and* $f_n \to f, a.e. x \in X$*, and* $\{f_n\}_{n=1}^{\infty}$ *satisfied a controlling condition, i.e. we can find a integrable function* $g \in L^1(X)$*, such that* $|f_n(x)| \leq |g(x)|, a.e. x \in X, \forall n \in \mathbb{N}^*$*, then we know,*

$$\lim_{n\to\infty} \int_X f_n(x) d\mu \to \int f(x) d\mu \tag{1}$$

In fact we have even stronger,

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu = 0 \tag{2}$$

This is a standard theorem in real analysis, we give the proof.

Proof: f is the point-wise limit of f_n so we know f is measurable and also dominate by g, so by triangle inequality we have:

$$|f - f_n| \leq 2|g|$$

Then the 1 is trivially true, due to a diagonal taking subsequences trick. For more subtle result 2, we need use reverse Fatou theorem to show it is true, roughly speaking we have,

$$\limsup_{n\to\infty} \int_X |f_n - f| \leq \int_X \limsup_{n\to\infty} |f_n - f| = 0$$

The key point is the first inequality above used the reverse Fatou theorem.

□

Now we discuss of the main result of the approximation identity. So first we need to define what is a approximation identity. a key ingredient is scaling. i.e. we given a function Φ and consider $\Phi_t = t^{-n}\Phi(\frac{x}{t})$, and we wish,

$$\lim_{t\to 0}(f * \Phi_t(x)) = f(x), for a.e. x \ \in \mathbb{R}^n \tag{3}$$

Whenever $f \in L^p, 1 \le p \le \infty$, but there need some technique assume to make this intuition to be tight, this lead the following definition.

Definition 2 (Approximation of the identity) *Suppose Φ is a fixed function on $\mathbb{R}^n$ that is appropriated small at infinity (have good enough decay rate), for example, take,*

$$|\Phi(x)| \le A(1 + |x|)^{-n-\epsilon} \tag{4}$$

Then we define $\{\Phi_t : \Phi_t(x) = t^{-n}\Phi(\frac{x}{t})\}$ to be an approximation of the identity.

The key theorem is the following, related the approximation of the indentity with the maximal operator.

Theorem 3

$$\sup_{t>0} |(\Phi_t * f)(x)| \le c_\Phi Mf(x) \tag{5}$$

For heat kernel, the thing is more subtle.

Theorem 4 *[Heat kernel estimate]*

$$\|f - e^{t\Delta} f\|_2 \le \|\nabla f\|_2 \sqrt{t} \tag{6}$$

Remark 5 *I know this theorem from Lieb's book. The power of 4 combine with Plancherel theorem could use to establish the Sobolev inequality, at least for the index $p = 2$.*

There are 3 ingredients which cold be useful.

1. the power of Rearrangement inequality involve in the Approximation of indentity operator. we could consider the relationship between $f*\Phi_t$ and $f*\overline{\Phi}_t$, where $\overline{\Phi}$ is constructed by take the average of Φ on the level set but the foliation of scaling. Intuition seems some monotonic property natural emerge.

2. There is a discretization model, i.e. the toy model on gragh, or we think it as correlation between particles, the key point is the rescaling deformation could be instead by semi group or renormalization property.

3. We consider the more general case, now there is not only one Φ but a group of them, i.e. $\Phi_k, k \in A$, this will involve some amenable theory I think.

We give two of the original and most important examples, First, if

$$\Phi(x) = c_n(1+|x|^2)^{\frac{-(n+1)}{2}}$$

where

$$c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

then $\Phi_t(x)$ is the possion kernel, and,

$$u(x,t) = (f * \Phi_t)(x)$$

Gives the solution of the Dirichlet problem for the upper half space,

$$\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$$

Namely

$$\Delta u = (\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2})u(x,t) = 0, \ \ u(x,0) \equiv f(x) \tag{7}$$

The second example is the Gaussian kernel,

$$\Phi(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}.$$

This time, if $u(x,t) = (f * \Phi_{t^{\frac{1}{2}}})(x)$, then u is a solution of the heat equation,

$$(\frac{\partial}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2})u(x,t) = 0, \ \ u(x,0) \equiv f(x) \tag{8}$$

3 Singular integral with L^2 bounded translation invariant operator

The main result proved in last note about singular integral is a conditional one, guaranteeing the boundedness on L^p for a range $1 < p \leq q$, on the assupution that the boundedness on L^q is already known; the most important instance of this occurs when $q = 2$. In keeping with this, we consider bounded linear transformation T from $L^2(\mathbb{R}^n)$ to itself that commute with translation. As is well known, such operator are characterized by the existence of a bounded function m on $\mathbb{R}^n$ (the "multiper"), so that T can be realized as,

$$\widehat{Tf(\xi)} = m(\xi)\widehat{f}(\xi) \tag{9}$$

Where $\widehat{}$ denotes the Fourier transform. Alternatively, at least on test function $f \in S$, T can be realized in terms of convolution with a kernel K,

$$Tf = f * K \tag{10}$$

Where K is the distribution given by $\hat{K} = m$. We shall now examine how the theorem with condition on singular integral weill lead to some result of this type of operator. Roughly speaking, it is due to now we know the boundedness on L^2, for technique condition, we need to assume the distribution K agree away from the origin with a function that is locally integrable away from the origin with a function that is locally integrable away from the origin; in this case we define the function by $K(x)$. Then 10 implies that,

$$Tf(x) = \int K(x-y)f(y)dy, \ for \ a.e. x \notin suppf. \tag{11}$$

Whenever f is in L^2 and f has campact support. Tis is the representation of singular integral in the present context. Next, the crucial hormander condition is then equivalent with,

$$\int_{|x| \geq c|y|} |K(x-y) - K(x)|dx \leq A \tag{12}$$

for all $y \neq 0$, where $c > 1$. In this case, the condition 12 have a further understanding, in fact,

Lemma 6

$$|(\frac{\partial}{\partial x}^{\alpha} K(x))| \leq A_\alpha |x|^{-n-|\alpha|}, \ for \ all \quad \alpha \tag{13}$$

or its weaker form, (here $\gamma > 0$ is fixed)

$$|K(x-y) - K()| \leq A\frac{|y|^\gamma}{|x|^{n+\gamma}}, whenever \ |x| \geq c|y|. \tag{14}$$

imply the hormander condition 12

PROOF: Integral by part. □

So, now the key point is how do K, satisfied such conditions, come about? It turns out that, toughly speaking, such condition on K have equivalent versions when sated in terms of the Fourier transform of K, namely the multiper m. This is transform the difficulties from physics space to fractional space In the future note, we will find a proof of the following Theorem:

Theorem 7 *For $m = \hat{K}$.*

If we assume that,

$$|(\frac{\partial}{\partial \xi}^{\alpha} m(\xi))| \leq A'_\alpha |\xi|^{-n-|\alpha|}, \ for\ all \ \ \alpha \tag{15}$$

holds for all α, then K satisfied 6 for all α.

If we assume that m satisfied the above inequality for all $0 \leq |\alpha| \leq l$, where l is the smallest integer $> \frac{n}{2}$, then K satisfied 12

Remark 8 *The multiplier m satisfied the second part condition of 7, are called Marcinkiewicz mulltiplier.*

4 Maximal function, singular integral, and square functions.