problem

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1 Original Problem

on P_Q^1,for $(a, b, c, d) \in (Z^{\times})^4$,$a + b + c + d = 0$,$\exists$ 1-forms w:
w has 4 single singular points with a double singular point...........(*)
the answer is (*) $\Leftrightarrow$ $abcd$ is a square number.

2 Some Naive Calculation

$\frac{w}{dz} = \frac{a}{z} + \frac{b}{z-1} + \frac{c}{z-\lambda} = \frac{g(z)}{z(z-1)(z-\lambda)}$
$g(z) = -dz^2 - z(a\lambda + b\lambda + a + c) + a\lambda$
$\Delta_g = (a + b)^2\lambda^2 = [2(a + c)(a + b) + 4ad]\lambda] + (a + c)^2 = 0$
so $\exists\lambda \in Q - 0, 1 \Leftrightarrow \Delta_{\Delta_g}$ is square number $\Leftrightarrow$ $abcd$ is square number.

3 The Statement Of Theorem

theorem 1 *for a 1 form in P_Q^1 in the form $\frac{w}{dz} = \frac{a}{z} + \frac{b}{z-1} + \frac{c}{z-\lambda} = \frac{g(z)}{z(z-1)(z-\lambda)}$,the form has 4 single singular points,we claim,the form has one double zero points iff,*

$$\exists a, b, c, d, t \in Q^{\times}, s.t : a + b + c + d = 0, abcd = t^2$$

remark 1 *there is a difficult is that in this way we can not determine all 1-form in P_Q^1 with 4 single singular points and one double zero points,because the is a fraction linear transformation.*

proof 1 *in fact,form $\frac{w}{dz}$ has a double zero point iff g(z) has a double zero points in $Q - \{0, 1\}$.*
case1:
we first assume that the solution of $g(z)$ never be 1 or 0.in this case,g(z) has

a double zero points $\Leftrightarrow \Delta_g = 0$.
calculation $g(z) = (a+b+c)z^2 - (a\lambda + b\lambda + a + c)z + a\lambda$,

$\Delta_g(z) = (a\lambda + b\lambda + a + c)^2 - 4(a+b+c)a\lambda = (a+b)^2\lambda^2 + [2(a+b)(a+c) + 4ad]\lambda + (a+c)^2$.
so (a,b,c,d) is the series of a 1-form in the form $\frac{w}{dz} = \frac{a}{z} + \frac{b}{z-1} + \frac{c}{z-\lambda} = \frac{g(z)}{z(z-1)(z-\lambda)}$ *iff* $(a+b)^2\lambda^2 + [2(a+b)(a+c)+4ad]\lambda + (a+c)^2$ *have a solution in* Q.
this equal to $\Delta_{\Delta_g} = t^2, \exists t \in Q$.
calculation $\Delta_{\Delta_g} = [2(a+b)(a+c)+4ad]^2 - 4(a+b)^2(a+c)^2 = 16(a+b)(a+c)ad + 16a^2d^2 = 16ad\{[a(a+b+c)+bc]+ad\} = 16abcd$.
so this equal to $\exists a,b,c,d,t \in Q, s.t: a+b+c+d=0, abcd = t^2$.
case2:
the solution of $g(z)$ *is 0 or 1. if* $g(0) = 0$*,a=0,if* $g(1) = 0$*,b=0.this is contraction with:*$a,b,c,d,t \in Q^\times$.
Q.E.D

remark 2 *we can assume (a,b,c,d) is primitive,this is just change the 1-form by a scaling.*

4 Solve The Diophantine Equation

theorem 2 *the primitive solutions* $(x,y,z,w) \in (Q^\times)^4$ *of equation*

$$\begin{cases} x+y+z+w=0 \\ xyzw = t^2 \end{cases}$$

is given by $(a,b,c,d,k_1,k_2,k_3,k_4,k_5,k_6)$ *,*$a,d,c,d,k_1,k_2,k_3,k_4,k_5,k_6 \in Q^\times$*,satisfied,*

$$k_1k_2k_3a^2 + k_1k_4k_5b^2 = k_2k_4k_6c^2 + k_3k_5k_6d^2 ...(*)$$

by the relation,

$$\begin{cases} x = k_1k_2k_3a^2 \\ y = k_1k_4k_5b^2 \\ z = k_2k_4k_6c^2 \\ w = k_3k_5k_6d^2 \end{cases}$$

proof 2 *the proof is trivial*

5 the structure of solution space of (*)

theorem 3 *$a^2+b^2=n$...(**),$a,b,n\in Q^\times$.$n=2^l p_1^{w_1} p^{w_2}...p_n^{w_n} q_1^{t_1}...q_m^{t_m}$,then (**) exists solution iff $\forall i, 2|q_i$,and in this case the number of solutions is $4\prod_{i=1}^{n}(1+w_i)$.*

proof 3 *the proof use a little of algebra number theory.*
$a^2+b^2=n \Leftrightarrow n=(a+bi)(a-bi)=N(c), c\in Q[i]$.
noticed $(a+bi,a-bi)=(2a,a-bi)=(a,b)$,we know prime in Z is prime in $Z[i]$ iff in the form $4k+3$.so if the solution of (**) exist,we must have $\forall i, 2|q_i$. on the other hand we know $a^2+b^2=p$ p has the form $4k+1$ has unique solution(if do not consider the permutation) so it is also easy to proof $a^2+b^2=p^k$ has $k+1$ solutions.in fact there is a unique solution with the condition $(a+bi,a-bi)=p^s, 0\le s\le k$.
Q.E.D

theorem 4 *$a^2+\lambda b^2=c^2+\lambda d^2$ ...(***),$a,b,n,\lambda\in Q^\times$.*

proof 4 *algebra number theory*

theorem 5 *the solution of:*
$k_1k_2k_3a^2+k_1k_4k_5b^2=k_2k_4k_6c^2+k_3k_5k_6d^2$,$a,d,c,d,k_1,k_2,k_3,k_4,k_5,k_6\in Q^\times$ have 15 kind of expression in parameter.12 of they are in the form of,

$$(k_1,k_2,k_3,k_4,k_5,k_6)=(k_4k_6c^2+u,-k_4k_5b^2+(\frac{d^2}{a^2}k_5k_6+\frac{k_4^2k_5k_6c^2b^2}{k_3^2a^2})\frac{1}{u},k_3,k_4,k_5,k_6)$$

3 of they are in the form of,

$$(k_1,k_2,k_3,k_4,k_5,k_6)=(\lambda,k_2,k_3,k_4,k_5,\frac{k_2k_3a^2+k_4k_5b^2}{k_2k_4c^2+k_3k_5d^2}\lambda)$$

proof 5 *this is obvious.*

theorem 6 *all solution of $k_1k_2k_3a^2+k_1k_4k_5b^2=k_2k_4k_6c^2+k_3k_5k_6d^2$, $a,d,c,d,k_1,k_2,k_3,k_4,k_5,k_6\in Q^\times$*
can be generated by (1,1,1,1,1,1) under 15 actions.

proof 6 *in fact if we want to get a solution $(k_1,k_2,k_3,k_4,k_5,k_6)$, under the action,we have,*
$(1,1,1,1,1,1)\mapsto(k_1,\overline{k}_2,1,1,1,1)\mapsto(k_1,k_2,\overline{k}_3,1,1,1)\mapsto(k_1,k_2,k_3,\overline{k}_4,1,1)\mapsto(k_1,k_2,k_3,k_4,\overline{k}_5,1)\mapsto(k_1,k_2,k_3,k_4,k_5,k_6)$. *Q.E.D*

6 the strategy to find solution of (*)

the results over seems do not enough to find all the solutions of (*)
we follow this strategy:
first we know fix (a, b, c, d),we can solve the solutions of $(k_1, k_2, k_3, k_4, k_5, k_6)$ in parameter.we can change $(k_1, k_2, k_3, k_4, k_5, k_6)$ into $(1, 1, 1, 1, 1, k_6)$ by 15 kind of action.so $\exists\lambda \in Q^{\times}$,

$$a^2 + \lambda b^2 = c^2 + \lambda d^2$$

this is a restrict of (a, b, c, d) and equal to say,
to find $(a, b, c, d, k_1, k_2, k_3, k_4, k_5, k_6)$ satisfied (*) it is suffice to find a $\lambda \in Q^{\times}$ and $(a, b, c, d) s.t. a^2+\lambda b^2 = c^2+\lambda d^2$ and fix (a, b, c, d) to find $(k_1, k_2, k_3, k_4, k_5, k_6)$ and by this way we can find all solution of (*).
every step is easy to finish so in some sense we can say we solve the problem.